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Some Application of second-order epi-derivativse in terme of $\rho -$Housdoroff distance.

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□ ABSTRACT □

The purpose of this research is to extend some results introduced by Rockafellar [19] in finite-dimensioal spaces to general Banach space using the $\rho -$Housdoroff distance convergent instead of epigraphical convergent . These results are aplications to study the second-order epi-derivatives of function to classe C^2 and to study the second-order epi-derivatives of sum two convex function and to study the second-order epi-derivatives of Moreau-Yosida approximate function also to study of the second-order epi-derivatives of composition convex function with linear operator .

Keywords: epigrqhp , Frechet differentiable, Moreau-Yosida approximate, epi-derivativse, proto-derivative, ρ _ Hausdorff distance

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دراسة بعض تطبيقات المشتقات فوق البيانية من المرتبة الثانية باستخدام مفهوم مسافة ρ –هاوسدورف .

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□ ملخّص □

الهدف من هذا البحث هو تعميم بعض النتائج التي درسها الرياضي روكافولار [19] في فضاءات منتهية البعد إلى فضاءات باناخ عامة مستبدلاً مفهوم التقارب فوق البياني بمفهوم تقارب مسافة ρ –هاوسدوف وهذه النتائج هي تطبيقات لدراسة المشتق الثاني لدالة من الصف C^2 , لدراسة المشتق الثاني لمجموع دالتين إحداهما من الصف C^2 , لدراسة المشتق الثاني لدالة مورو– يوشيدا والعلاقة بين مشتق–بروتو للمؤثر الحال J_λ^f ومشتق بروتو للمؤثر الحال $J_\lambda^{f''}$ وأيضا لدراسة المشتق الثاني لتركيب دالة مع مؤثر خطي ......الخ.

الكلمات المفتاحية : فوق البياني , تقريب مورو–يوشيدا, المشتق –فوق البياني, مشتق –بروتو, مسافة ρ – هاوسدوف .

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Introduction:

During the last few years many works has been devoted to epigraphical analysis and their applications of optimization problems ,it is to study the functions by using the property of their epigraphs, and it is the introduction of original the new concepts: *epi-convergence, epi-distance, epi-derivative, epi-differentiable, epi- integral…..* This analysis is addressed naturally to study the minimization problems (see for example [1,2,3,7, 20]).

Epi-drivatives have many applictions in optimization as approached through nonsmooth analysis. In particular, second-order derivatives can be used to obtain optimality conditions and carry out in sensitivity analysis.Many authors have tried to define second-order derivatives in quite different ways. Most definitions have been confined to finite-valued function; see for example [8,9,11,120]) for nonconvex and [2,3,13,22]) for convex.

The main idea developed in this paper is to replace the Mosco- epi-convergence by the ρ -Hausdorff distance convergence, a concept introduced by Mosco too, but developed by many authors (see [4,5,6,7,8,9,19,20]), and which has proved to be efficient in the quantitative analysis of the stability of minimization problems in general Banach spaces.

This paper is organized as follows. In section 1, we give general introduction . In section 2, we fix the notations and recall some definitions and some known results concerned the second-order epi-derivative of convex function f and the proto-derivatives of set-valued mapping ∂f . In section 3,

we give the first main result (see proposition 3.2), concerned the second-order epi-derivative of the sum two functions, and we give the second important result (see Theorem 3.3); that is the cnnection between the second-order epi-derivative of convex function f and the second-order epi-derivative of the Moreau-Yosida approximate f_λ , we prove also that the mapping J_λ^f is proto- differentiable , we etablishe that $f \circ A$ is twice epi-differentiable at x relative to $A^* x^*$.

Notation and definitions:

Let us recall some definitions and notions, which are of common use in the context of convex analysis and optimization; for further information, we can refer to $[1,13,14]$. Let $(X, \|.\|)$ be a normed linear space and $(X^*, \|.\|_*)$ its dual, the duality pairing between $x^* \in X^*$ and $x \in X$ is denoted by $\langle x, x^* \rangle$, and let $f : X \to \overline{R}$ of the real valued extension function defined on X, we well denote the set of the real valued extended functions defined on X by $\overline{R}^X$. For a function $f \in \overline{R}^X$ the set :

$$epi\, f = \{ (x, \alpha) \in X \times R \ / \ f(x) \le \alpha \}$$

is called the ***epigraph*** of f , and f is called *convex* (*lower semiconti-nuous*) if its *epigraph* is a convex (closed) subset of $X \times R$. Furthermore, f is called *proper* if its epigraph nonempty.

Again, $\Gamma(X)$ will denote the proper, lower semi continuous convex functions defined on X , and dually, $\Gamma^*(X^*)$ will denote the proper, weak* lower semicontinuous convex functions defined on X^*. It is well known that to each nonempty closed convex subset C of X its *indicator function* $\delta(.,C) \in \Gamma(X)$, defined by the formula

$$\delta(.,C)=\begin{cases}0 & if\ x\in C\\ +\infty & if\ x\notin C\end{cases}$$

For $f\in\Gamma(X)$, its *conjugate* $f^*\in\Gamma^*(X^*)$ is defined by the familiar formula

$$f^*(x^*)=\sup_{x\in X}\left\{<x,x^*>-f(x)\right\}$$

The *subdifferential* of $f\in\overline{R}^X$ at x_0, denoted by $\partial f(x_0)$, is defined by :

$$\partial f(x_0)=\left\{x^*\in X^*/\ f(x)\geq f(x_0)+<x-x_o,\ x^*>;\ \forall x\in X\right\}$$

This set is convex (closed) if f is convex (lower semi continuous), and one has the following equivelent:

$x^*\in\partial f(x_0)\Leftrightarrow\exists\varepsilon>0,\forall x\in B(x,\varepsilon),\ f(x)\geq f(x_0)+<x^*,\ x-x_o>$

A Banach spaces X is said to be Uniformly convex*(U.C*, in short*)* if for each $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that whenever ;

$\|x\|\leq\|y\|\leq 1\ and\ \|x-y\|\geq\varepsilon,\ then\ \ \|x+y\|\leq 2(1-\delta(\varepsilon))$

A Banach spaces X is said to be Uniformly smooth*(U.S*, in short), if X^* is Uniformly convex.

A Banach spaces X is said to be **super-reflexive** if and only if , it is *U.C and* U.S .

Set-Convergence

- Given a sequence $\{C_n,C;n\in N\}$ of subsets of X, the τ-lower limit of the sequence $\{C_n;n\in N\}$ denoted by $\tau-\liminf_n C_n$ is the closed subset of X defined by:

$$\tau-\liminf_n C_n:=\left\{x\in X\ /\exists\ (x_n)_{n\in N};\ x_n\in C_n\ ;x_n\xrightarrow[n]{\tau}x\right\}$$

the τ-upper limit of the sequence $\{C_n;n\in N\}$ denoted by $\tau-\limsup_n C_n$ is the closed subset of X defined by:

$$\tau-\limsup_n C_n:=\left\{x\in X\ /\exists(n_k)_{k\in N};\exists(x_k)_{k\in N}\ \forall k\in N;x_k\in C_{n_k};x_k\xrightarrow[k]{\tau}x\right\}$$

the sequence $\{C_n;n\in N\}$ is said to be ***Kuratowski-painlevé*** convergent to C for the topology τ, or briefly τ-convergent, if the following conclisions hold:

$$\tau-\limsup_n C_n\subseteq C\subseteq\tau-\liminf_n C_n$$

We denoted by $C=\tau-\lim_n C_n$, is the closed subset of X

- • When X is a reflexive Banach space and the sets are closed and convex, the sequence $\{C_n;n\in N\}$ is said to ***Mosco- convergent*** to C, denoted by

$C=M-\lim_n C_n$,

if $$s-\limsup_n C_n\subseteq C\subseteq w-\liminf_n C_n$$

where s (resp. w) is the strong (resp.weak) topologies on X .

Epigraphical convergence [1]

Let $\left\{ f_n, f : X \rightarrow \overline{R} \;;\; n \in N \right\}$ be a sequence of extended real valued functions. If the sequence $\{epi\, f_n \,;\, n \in N\}$ is Kuratowski-painlevé convergent to $epi\, f$ in $X \times R$ for the product topology; then we say that the sequence $\{f_n \;;\, n \in N\}$ epi-convergent to f and we write

$$f = epi - \lim_n f_n \,.$$

This is equivalent to say that, for any $x \in X$, the two following statements hold :

i) for any a sequence $\left(x_n\right)_{n\in N}$; $x_n \xrightarrow[n]{\tau} x \;/\; f(x) \le \liminf_n f_n(x_n)$.

ii) there exists sequence $(\zeta_n)_{n\in N}$; $\zeta_n \xrightarrow[n]{\tau} x \;/\; f(x) \ge \limsup_n f_n(\zeta_n)$.

Mosco-epigraphical convergence . [15]

Let X be a reflexive Banach space and $\{f_n, f \;;\, n \in N\}$ be a sequence of functions to $\Gamma(X)$. We say that f is the Mosco-epi-limite of the sequence $\{f_n \;;\, n \in N\}$ and we write $f = M - epi - \lim_n f_n$,If the sequence $\{epi\, f_n \,;\, n \in N\}$ Mosco- convergent to $epi\, f$.

This is equivalent to say that, for any $x \in X$, the two following statements hold :

i) for any sequence $\left(x_n\right)_{n\in N}$; $x_n \xrightarrow[n]{w} x \;/\; f(x) \le \liminf_n f_n(x_n)$

ii) there exists sequence $(\zeta_n)_{n\in N}$; $\zeta_n \xrightarrow[n]{s} x \;/\; f(x) \ge \limsup_n f_n(\zeta_n)$.

ρ-Housdoroff distance*s* [3]

For all subset C of X , we denote the distance from some point x in X to C by :

$$d\,(x\,,C\,) = \inf_{y\in C} \|x - y\| \quad ; \quad (\, if \quad C = \varnothing, \quad d(x\,,C\,) = +\infty\,)$$

For each $\rho \ge 0$, ρB denotes the closed ball of radius ρ ; and for any subset C of X , we define C_ρ by

$$C_\rho := C \cap \rho B$$

For any pair C and D of subsets of X , the Housdoroff excess of C over D is defined by :

$$e\,(\,C\,,D\,) := \sup_{x\in C} d(x,D) \quad ; \quad (\, e(C,D) = 0 \;,\; if \;\; C = \varnothing)$$

and for all $\rho \ge 0$, the $\rho-$Housdoroff distance*s* between C and D is defined by :

$$haus_\rho(\,D\,,C\,) = sup\left\{\, e(C_\rho\,,\; D\,)\;,\; e(D_\rho\,,\; C\;)\,\right\}$$

A sequence of subsets $(D_n)_{n\in N}$ of X , is said to converge with respect to the $\rho-$ Housdoroff distance*s* to some D iff for all $\rho \ge 0$,

$$\lim_{n\to\infty} haus_\rho\,(\,D_n\;,\; D\,) = 0$$

This means that for each $\rho \ge 0$ and each $\varepsilon \ge 0$, for n large enough the following inclusios hold

$$C \cap \rho B \subset C_n + \varepsilon B \quad and \quad C_n \cap \rho B \subset C + \varepsilon B \;.$$

Clearly, all the above notions make sense in a general normed space X . When X is a reflexive Banach space and the sets are closed and convex

$$\lim_{n\to\infty} haus_\rho(D_n , D) = 0 \quad \text{for each } \rho \geq 0 \quad \Rightarrow \quad D = M - \lim_{n\to+\infty} D_n .$$

ρ-Housdoroff distances on $\overline{R}^X$ [3]

a) For all $\rho \geq 0$, the $\rho-$Housdoroff distance*s* between two functions f , $g \in \overline{R}^X$ is defined by :

$$h_\rho(f, g) = haus_\rho(epi\ f\ ,\ epi\ g)$$

where $epi\ f\ and\ epi\ g$ are two subsets of $X \times R$, and the ball of $X \times R$ is the set :

$$\rho B_{X\times R} = \left\{ (x, \alpha) \in X\times R \ / \ \|x\| \leq \rho \ , \ |\alpha| \leq \rho \right\}$$

b) A sequence of functions $(f_n)_{n\in N}$ of $\overline{R}^X$, is said to converge with respect to the $\rho-$Housdoroff distance*s* to some f iff for all $\rho \geq 0$, $\lim_{n\to\infty} h_\rho (f_n , f) = 0$

We write : $f_n \xrightarrow{h_\rho} f$ or $f = (h_\rho - epi) - \lim_{n\to+\infty} f_n$.

We recall two fundamental results, the first gives the bicontinuity between the functions of $\Gamma(X)$ and it conjugates of $\Gamma^*(X^*)$, and the second gives the continuity of the sum functions in $\Gamma(X)$, with respect to the $\rho-$Housdoroff epigraphical distance .

ρ-Housdoroff graphical distances:

Let Y a general normed space, Given an operator $A: X \rightrightarrows Y$, possibly multivalued, its graph is denoted by:

$$gphA = \left\{ (x, y) \in X\times Y;\ y\in A(x) \right\}$$

(a) For all $\rho \geq 0$, the $\rho-$Housdoroff graphical distance*s* between two operators $A: X \rightrightarrows Y$ and $B: X \rightrightarrows Y$ is defined by :

$$haus_\rho(A, B) = haus_\rho(gph\,A,\ gph\,B)$$

where $gph\,A\ and\ gph\,B$ are two subsets of $X \times Y$, and the ball of $X \times Y$ is the set :

$$\rho B_{X\times Y} = \left\{ (x, y) \in X\times Y \ / \ \|x\| \leq \rho \ , \ \|y\| \leq \rho \right\}$$

(b) A sequence of operators $(A_n)_{n\in N}$, is said to converge with respect to the $\rho-$Housdoroff graphical distance*s* to some A iff for all $\rho \geq 0$,

$$\lim_{n\to\infty} haus_\rho (A_n , A) = 0$$

The concept $\rho-$Housdoroff distances on $\overline{R}^X$ is also called the

ρ - Housdoroff epigraphical distance introduced in $[5,6]$ and has been developed by many authors in various field $[4,5,6,,20,21]$.

Proposition 2.1 [6]

Let $\{ f_n , f ; n \in N \}$ be a sequence of functions in $\Gamma(X)$. Then for all $\rho \geq 0$, we have the following equivalent :

$$f_n \xrightarrow{h_\rho} f \Leftrightarrow f_n^* \xrightarrow{h_\rho} f^*$$

Theorem 2.2 [4]

Let X be a Banach space. For any sequence $\{ f_n , f ; n \in N \}$ closed proper convex functions, the following implication holds : $(i) \Rightarrow (ii)$ where

$$(i)\ f = (h_\rho - epi) - \lim_n f_n$$

$$(ii)\ \partial f = (h_\rho - gph) - \lim_{n \to +\infty} \partial f_n + N.C$$

If X is super-reflexive, then the converse implication holds, that is $(i) \Leftrightarrow (ii)$.

$N.C \equiv \exists(\xi,\eta) \in gph\partial f, \exists(\xi_n,\eta_n) \in gph\partial f_n \text{ such that} (\xi_n,\eta_n) \to (\xi,\eta) \text{ and } f_n(\xi_n) \to f(\xi)$

• What we will be dealing with is a family of functios $(\varphi_t)_{t>0}$ parameterized by $t > 0$. The $\rho-$Housdoroff distances convergence of φ_t to φ as $t \downarrow 0$ is defined in a natural way by saying $\varphi_{t_n} \xrightarrow{h_\rho} \varphi$ for every sequence $t_n \downarrow 0$. i.e

$$\lim_{n \to \infty} h_\rho (\varphi_{t_n} , \varphi) = 0 , \forall \rho \geq 0.$$

The following proposition is immediate :

Proposition 2.3

Let $\varphi_n \xrightarrow{h_\rho} \varphi$. If $(\varphi_n)_{t>0}$ are closed convex functions, then so is φ.

second-order epi- derivative [16,19,21]

Let $f : X \to \overline{R}$ be finite at $x \in X$. Let $x^* \in X^*$ and consider the second-order difference quotient functions :

$$\varphi^f_{t,x,x^*}(\xi) = \frac{1}{t^2}\{f(x+t\xi) - f(x) - t < x^*, x >\}; \xi \in X. \quad (t > 0)$$

If these functions are $\rho-$Housdoroff epigraphical distance -convergent (as $(t \downarrow 0)$ to some function φ having $\varphi(0) \neq -\infty$, then we say that f is twice epi- differentiable at x relative to x^*, and φ is called the second-order epi- derivative of f at x relative to x^*. We then write f''_{x,x^*} instead of φ, i.e

$$f''_{x,x^*} = (h_\rho - epi) - \lim_{t \downarrow 0} \varphi^f_{t,x,x^*}.$$

In termes of sequences ,

$$f''_{x,x^*} = (h_\rho - epi) - \lim_{n \to +\infty} \varphi^f_{t_n,x,x^*}, \ \forall t_n \downarrow 0$$

Some elementary properties entailed by these defintions are explored in the following propositions.

Proposition 2.4. [21]

The second-order epi-derivative function f''_{x,x^*} , if it exists, is lower semicontinuous, proper convex, positive homogenity of degree 2, $f''_{x,x^*} \geq 0$, $f''_{x,x^*}(0) = 0$ and 0 is minimal point of f''_{x,x^*} , i.e $0 \in \partial f''_{x,x^*}(0)$.

Theorem2.5 (Conjugacy). [21]

Let $f : X \to \overline{R}$ be a closed proper convexe function.Then one has

f is twice epi- differentiable at x relative to x^* if and only if f^* is twice epi - differentiable at x^* relative to x . More precisely we have :

$\left(f''_{x,x^*}\right)^* = \left(f^*_{x^*,x}\right)''$.

proto- derivative [18]

Given a multifunction $\Gamma : X \rightrightarrows Y$, a point $x \in X$ with $\Gamma(x) \neq \phi$ and a point $y \in \Gamma(x)$. We consider the difference quotient multifunctions :

$$D^{\Gamma}_{t,x,y}(\xi) = \frac{1}{t}\{\Gamma(x + t\xi) - y\};\ \xi \in X . \quad (t > 0)$$

If these multifunctions are $\rho -$Housdoroff graphical distance -convergent (as $(t \downarrow 0)$ to some multifunction D , then we say that Γ is proto- differentiable at x relative to y , and D is called proto- derivative of Γ at x relative to y . We then write $\Gamma'_{x,y}$ instead of A , i.e

$\Gamma'_{x,y} = \left(H_\rho - gph\right) - \lim_{t \downarrow 0} D^{\Gamma}_{t,x,y}$

In termes of sequences ,

$\Gamma'_{x,y} = \left(H_\rho - gph\right) - \lim_{n \to +\infty} D^{\Gamma}_{t_n,x,y}$, $\forall\, t_n \downarrow 0$.

Some elementary consequences of the definition of proto – differentiabliliy by :

Proposition 2.6

Let $\Gamma : X \rightrightarrows Y$ be proto- differentiable at x relative to y , where $y \in \Gamma(x)$. Then proto- derivative $\Gamma'_{x,y}$ has closed graph and satsisfies:

$0 \in \Gamma'_{x,y}(0), \quad and \qquad \Gamma'_{x,y}(\lambda\xi) = \lambda^2 \Gamma'_{x,y}(\xi)$

for all $\xi \in X \quad and \ \lambda > 0$.

Proof: The verification as in the finite- dimensional case (see [18, *proposition* 2.4]) .

Thorem 2.7 [20]

Let $f : X \to \overline{R}$ be a closed proper convexe function, $x \in X$ such that $f(x)$ is finite and $x^* \in X^*$. We consider the tow folloing statements :

(a) f is twice epi- differentiable at x relative to x^* .

(b) $x^* \in \partial f(x)$ and ∂f is proto- differentiable at x relative to x^* .

Then (a) $\Rightarrow$ (b). If X is super- reflexive, we have (a) $\Leftrightarrow$ (b), and the proto - derivative of ∂f at x relative to x^* is the subdifferential of f''_{x,x^*} . More precisely,

$$\partial\left(f''_{x,x^*}\right) = (\partial f)'_{x,x^*} .$$

Results and Discussion:

Proposition 3.1 Let $f : X \to \overline{R}$ be C^2 convexe function in a neighborhood of $x \in X$. Then f is twice epi-differentiable at x and the the second-order epi-derivatve $f''_{x,Df(x)}$ is given by

$$f''_{x,Df(x)}(\xi) = \frac{1}{2} < D^2 f(x)\xi, \xi > \quad (3.1)$$

Proof. By Taylor's fourmula, one can write

$$f(x+t\xi) = f(x) + t < Df(x)\xi, \xi > + < D^2 f(x)\xi, \xi > + \|\xi\|^2 \theta(t\xi)$$

where $\lim_{t\downarrow 0} \theta(t\xi) = 0$. Let

$$\varphi_t(\xi) := \varphi^f_{x,Df(x)}(\xi) = \frac{1}{2} < D^2 f(x)\xi, \xi > + \|\xi\|^2 \theta(t\xi)$$

$$\varphi(\xi) := \frac{1}{2} < D^2 f(x)\xi, \xi >, \quad and \quad \theta_t(\xi) := \|\xi\|^2 \theta(t\xi)$$

It is easly to see that φ is convex function and $\varphi(0) = 0$. Since $0 \in \text{int}\, dom\,\varphi$, and from the result $[5, corollary\ 2.9]$, we have :

For all $\rho \geq 0$, there is $\rho_1 \geq 0$ and $k(\rho)$ such that

$$0 \leq haus_\rho(\varphi_t, \varphi) = haus_\rho(\varphi + \theta_t, \varphi)$$
$$\leq k(\rho)\max\left\{haus_{\rho_1}(\varphi, \varphi);\ haus_\rho(\theta_t, 0)\right\}$$

Hence for all $t_n \downarrow 0$, one has $\lim_{n\to+\infty} haus_\rho\left(\varphi_{t_n}, \varphi\right) = 0$ i.e f is twice epi-differentiable at x relative to $Df(x)$, and the the second-order epi- derivatve is given by (3.1).

Proposition 3.2 : Suppose f, g are closed proper convexe functions on X and f is C^2 in a neighborhood of $x \in X$. Let $h = f + g$. Then

$$y^* = Df(x) + x^* \in \partial h(x) \quad \Leftrightarrow \quad x^* \in \partial g(x) \quad (3.2)$$

If g is twice epi- differentiable at x relative to x^* and $\text{int}\left(dom\, g''_{x,x^*}\right) \neq \phi$, then

(a) h is twice epi- differentiable at x relative to y^*, and

$$h''_{x,y^*}(\xi) = \frac{1}{2} < D^2 f(x)\xi, \xi > + g''_{x,x^*}(\xi) \quad (3.3)$$

(b) ∂h is proto- differentiable at x relative to y^*, and

$$(\partial h)'_{x,y^*} = D^2 f(x) + (\partial g)'_{x,x^*} \quad (3.4)$$

Proof :

The equivalence (3.2) is a well-known fact in convex analysis :

$\partial h(x) = \partial f(x) + \partial g(x), \quad x \in X$

To prove (a) of the proposition and (3.3), let $t_n \downarrow 0$ and $\xi \in X$

Let φ_t^h , φ_t^f , φ_t^g be the difference quotients of h (at x relative to y^*), f (at x^* relative to $Df(x)$), g (at x relative to x^*), respectively.

one has

$\varphi_t^h(\xi) = \varphi_t^f(\xi) + \varphi_t^g(\xi), \ \xi \in X$

Since f is C^2, one has (see proposition 3.1)

For all $\rho \geq 0$

$h_\rho\left(\varphi_{t_n}^f, f''_{x,Df(x)}\right) \xrightarrow[n]{} 0$

Since g is twice epi-differentiable at x relative to y^*, then for all $\rho \geq 0$ one has

$h_\rho\left(\varphi_{t_n}^g, g''_{x,y^*}\right) \xrightarrow[n]{} 0$

On the other hand, since $\operatorname{int}\left(dom g''_{x,x^*}\right) \neq \phi$ and $f''_{x,Df(x)}$ is evry where difinit , we can suppose that $0 \in \operatorname{int}\left(dom g''_{x,x^*} - dom f''_{x,Df(x)}\right)$ and we apply $[5, corollaily\ 2.9]$, to get for all $\rho \geq 0$:

$$h_\rho\left(\varphi_{t_n}^f + \varphi_{t_n}^g,\ f''_{x,Df(x)} + g''_{x,y^*}\right) \xrightarrow[n]{} 0 \qquad (3.5)$$

Hence $h_\rho\left(\varphi_{t_n}^h,\ h''_{x,y^*}\right) \xrightarrow[n]{} 0$, with

$$h''_{x,y^*}(\xi) = f''_{x,Df(x)} + g''_{x,x^*}(\xi) \ ; \quad y^* = Df(x) + x^* \qquad (3.6)$$

From (3.1) and (3.6) , we have (3.3) , and this proves (a).

To prove (b) of the proposition and (3.4) , let $t_n \downarrow 0$ and $\xi \in X$ Let $\Phi_t^{\partial h}$, Γ_t^{Df} , $\Delta_t^{\partial g}$ be the difference quotients of ∂h (at x relative to y^*), Df (at x^* relative to $Df(x)$), ∂g (at x relative to x^*), respectively.

one has

$\Phi_t^{\partial h}(\xi) = \Gamma_t^{Df}(\xi) + \Delta_t^{\partial g}(\xi), \quad \xi \in X$

From (3.5) and we apply $[5, theorem\ 3.5]$, to get for all $\rho \geq 0$ and $t_n \downarrow 0$:

$$haus_\rho\left(\partial\varphi_{t_n}^f + \partial\varphi_{t_n}^g,\ \partial f''_{x,Df(x)} + \partial g''_{x,y^*}\right) \xrightarrow[n]{} 0$$

$$haus_\rho\left(\Gamma_{t_n}^{\partial f} + \Delta_{t_n}^{\partial g},\ \partial f''_{x,Df(x)} + \partial g''_{x,y^*}\right) \xrightarrow[n]{} 0$$

$$haus_\rho\left(\Phi_{t_n}^{\partial h},\ \partial f''_{x,Df(x)} + \partial g''_{x,y^*}\right) \xrightarrow[n]{} 0$$

From (a) of theorem 2.7 we have

$$\partial\left(f''_{x,Df(x)}\right) = (Df)'_{x,Df(x)} = D^2 f(x) \quad \text{and} \quad \partial\left(g''_{x,x^*}\right) = (\partial g)'_{x,x^*}$$

Thus $haus_\rho\left(\Phi_{t_n}^{\partial h},\ D^2 f(x) + (\partial g)'_{x,y^*}\right) \xrightarrow[n]{} 0$

Hence $\left(\partial h\right)'_{x,y^*} = D^2 f\left(x\right) + \left(\partial g\right)'_{x,x^*}$ and this proves (b) . ■

Thorem 3.3 (*Moreau-Yosida approximate*).

Let X be a Hilbert space and f be a closed proper convexe function on X . Let $\lambda > 0$. Then the Moreau-Yosida approximate

$$f_\lambda\left(x\right) = \inf_{u\in X}\left\{ f\left(u\right) + \frac{1}{2\lambda}\left\|x-u\right\|^2 \right\} \tag{3.6}$$

is a C^1 function. The infimum above is always attained at a unique point which will be denoted by $J_\lambda^f\left(x\right)$. The mapping J_λ^f is continuous in x and the function Df_λ is Lipschitzian with constant $\left(\frac{1}{\lambda}\right)$. One has

$$u = J_\lambda^f\left(x\right) \Leftrightarrow \frac{1}{\lambda}\left(x-u\right)\in \partial f\left(x\right) \tag{3.7}$$

$$Df_\lambda\left(x\right) = \frac{1}{\lambda}\left(x-u\right) \tag{3.8}$$

with x,u and $z = Df_\lambda\left(x\right)$ as above , we have the following statements:

(a) If f is twice epi- differentiable at u relative to z, then f_λ is twice epi-differentiable at x relative to z and

$$\left(f_\lambda\right)''_{x,z} = \left(f''_{u,z}\right)_\lambda = f''_{u,z} \underset{e}{+} \frac{1}{2\lambda}\left\|.\right\|^2 \tag{3.9}$$

With $x = u + \lambda x^*,\ x^* \in X^*$.

(b) Undre the same assumption as in (a), the mapping J_λ^f is proto- differentiable at x relative to u and proto-derivative is give by :

$$\left(J_\lambda^f\right)'_{x,u}\left(\xi\right) = J_\lambda^{f''_{u,z}}\left(\xi\right) = \arg\min_{\eta\in X}\left\{ f''_{u,z} + \frac{1}{2\lambda}\left\|\eta-\xi\right\|^2 \right\} \tag{3.10}$$

Proof:

The properties of f_λ, J_λ^f and Df_λ are well-known for in facts in convex analysis; see $\left[1, Theorems\ 3.24\ and\ 3.56\right]$. We wish to show (a) and (3.9), we write (3.6) by the formul

$$f_\lambda = f \underset{e}{+} \frac{1}{2\lambda}\left\|.\right\|^2 \quad \text{(infimal convolution)}$$

So $$\left(f_\lambda\right)^* = f^* + \frac{\lambda}{2}\left\|.\right\|^2 \tag{3.11}$$

By theoreme 2.5, f is twice epi- differentiable at u relative to z if and only if f^* is twice epi- differentiable at z relative to u. Let $g = \frac{1}{2\lambda}\left\|.\right\|^2$ hence $g^* = \frac{\lambda}{2}\left\|.\right\|^2$ is C^2 and the Fréchet derivative of $Dg^*\left(x^*\right) = \lambda x^*$. Thus from (3.11) and proposition 3.1, $\left(f_\lambda\right)^*$

is twice epi- differentiable at z relative to x with $x = u + Dg^*\left(x^*\right)$. By theorem 2.5 again, f_λ is twice epi- differentiable at x relative to z and one has

$$\left[\left(f_\lambda\right)''_{x,z}\right]^* = \left(f_\lambda\right)^*_{z,x} \qquad (3.12)$$

From (3.11) and (3.3) , we have

$$\left(f_\lambda\right)^*_{z,x} = \left(f^*\right)''_{z,u} + \frac{\lambda}{2}\|\cdot\|^2 = \left(f''_{u,z}\right)^* + \frac{\lambda}{2}\|\cdot\|^2 \qquad (3.13)$$

Comparing (3.12) with (3.13), we conclu :

$$\left[\left(f_\lambda\right)''_{x,z}\right]^* = \left(f''_{u,z}\right)^* + \frac{\lambda}{2}\|\cdot\|^2 = \left(f''_{u,z} \underset{e}{+} \frac{1}{2\lambda}\|\cdot\|^2\right)^*$$

Thus

$$\left(f_\lambda\right)''_{x,z} = f''_{u,z} \underset{e}{+} \frac{1}{2\lambda}\|\cdot\|^2 = \left(f''_{u,z}\right)_\lambda$$

This shows (a) and (3.9).

To prove (b) of the Theorem and (3.10). Let

$$\Delta^{J^f_\lambda}_{t,\ x,u}(\xi) = \frac{1}{t}\left\{J^f_\lambda\left(x+t\xi\right) - u\right\};\ \xi \in X. \quad (t>0)$$

From (3.7) , we have $J^f_\lambda(x) = x - \lambda Df_\lambda(x)$, hence

$$\Delta^{J^f_\lambda}_{t,\ x,u}(\xi) = \frac{1}{t}\left\{x + t\xi - \lambda Df_\lambda\left(x+t\xi\right) - x + \lambda Df_\lambda(x)\right\};\ \xi \in X. \quad (t>0)$$

$$= \xi - \frac{1}{t}\lambda\left[Df_\lambda\left(x+t\xi\right) + Df_\lambda(x)\right];\ \xi \in X. \quad (t>0)$$

$$:= \xi - \Delta^{J^f_\lambda}_{t,\ x,z}(\xi)$$

Thus For all $\rho \geq 0$, J^f_λ is proto- differentiable at x relative to u if and only if Df_λ is proto- differentiable at x relative to $Df_\lambda x$ and its protp-derivative is givn by :

$$\left(J^f_\lambda\right)'_{x,u}(\xi) = \xi - \lambda\left(Df_\lambda\right)'_{x,z}(\xi). \qquad (3.14)$$

From (3.10), one has

$$\left(f_\lambda\right)''_{x,z} = \inf_{\eta\in X}\left\{f''_{x,z}(\eta) + \frac{1}{2\lambda}\|\eta - \xi\|^2\right\};\quad \xi \in X \qquad (3.15)$$

The infimum above is attained at a unique point which will be denoted by $J^{f''_{u,z}}_\lambda(\xi)$. Applying (a) to (3.15) with the function $f''_{u,z}$ in place of f , we gets,

$$J^{f''_{u,z}}_\lambda(\xi) = \xi - \lambda D\left(f_\lambda\right)''_{x,z}(\xi). \qquad (3.16)$$

By theorem 2.7, Df_λ is proto- differentiable at x relative to $Df_\lambda x$ and

its proto-derivative is the subdifferential (Fréchet derivative) of $\left(f_\lambda\right)''_{x,Df_\lambda(x)}$, therefore :

$$\left(Df_\lambda\right)'_{x,z}(\xi) = D\left(f_\lambda\right)''_{x,z}(\xi) \qquad (3.17)$$

Hence

$$J_{\lambda}^{f''_{u,z}}(\xi) = \xi - \lambda (Df_{\lambda})'_{x,z}(\xi). \qquad (3.18)$$

Comparing (3.17) with (3.12), one has (3.10). which completes the proof.■

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Corollary 3.4: Let C be a nonempty closed convex set in a Hilbert space X. Let P_C be the projection map on C . Suppose the indicator function δ_C is twice epi-differentiable at $u := P_C$ relative to $v := x - P_C(x)$, then P_C is proto-differentiable at x relative to v and its proto-derivative is continuous mapping obtained as the solution of the following problem :

$$(P)_{\xi} := Minimize\left\{ (\delta_C)''_{u,v}(\eta) + \frac{1}{2\lambda}\|\eta - \xi\|^2 ; \eta \in X \right\}$$

Proof: Just apply the above theorem with $f = \delta_C$ and $\lambda = 1$.

Proposition 3.5

Let Y be a Banch space, $A : X \to Y$ linear continuous mapping and $f : X \to \overline{R}$ be a closed proper convexe function.

If f is twice epi- differentiable at Ax relative to $x^* \in \partial f(Ax)$ and that

$$0 \in \operatorname{int}\left(R(A) - dom f''_{Ax,x^*}\right) \qquad (3.19)$$

Then $f \circ A$ is twice epi- differentiable at x relative to A^*x^* and

$$(f \circ A)''_{x,A^*x^*}(\xi) = \left(f''_{Ax,x^*} \circ A\right)(\xi) = f''_{Ax,x^*}(A\xi) \qquad (3.20)$$

Proof:

Let $\varphi_t^{f\circ A}$, φ_t^f be the difference quotients of $f \circ A$ (at x relative to A^*x^*), f (at Ax relative to x^*), respectively. Then for all $\xi \in X$

$$\begin{aligned} \varphi_t^{f\circ A}(\xi) &= \frac{1}{t^2}\{(f \circ A)(x + t\xi) - (f \circ A)(x) - t < A^*x^*, \xi >\} \\ &= \frac{1}{t^2}\{f(Ax + tA\xi) - f(Ax) - t < x^*, A\xi >\} \\ &= \varphi_t^f(A\xi) = (\varphi_t^f \circ A)(\xi). \end{aligned}$$

since f is twice epi- differentiable at Ax relative to x^*, then for all sequence $t_n \downarrow 0$ and for all $\rho \geq 0$, one has

$$h_{\rho}\left(\varphi_{t_n}^f, f''_{Ax,x^*}\right) \xrightarrow[n]{} 0 \qquad (3.21)$$

By (3.19) and appling $[13, corollary\, 2.6]$, we have :

For all $\rho \geq 0$, there is $\rho_1 \geq 0$ and $k(\rho)$ such that

$$0\leq h_\rho\left(\varphi_t^{f\circ A}, f''_{Ax,x*}\circ A\right)=h_\rho\left(\varphi_t^{f}\circ A,\, f''_{Ax,x*}\circ A\right)$$
$$\leq k(\rho)\, h_{\rho_1}\left(\varphi_t^{f}, f''_{Ax,x*}\right) \qquad (3.22)$$

Combining (3.22) with (3.21), we obtient for all $t_n\downarrow 0$ and for all $\rho\geq 0$ $\lim_{n\to+\infty} haus_\rho\left(\varphi_{t_n}^{f\circ A}, f''_{Ax,x^*}\circ A\right)=0$ i.e $f\circ A$ is twice epi- differentiable at Ax relative to x^*, and the second-order epi-derivatve is given by (3.20). which completes the proof.■

Conclusions and Recommendations:

In this paper,we presented some application of second-order epi-derivatives of Frechet differentiable convex function and Moreau-Yosida approximate function that plays an important role in nonsmooth analysis and in statements of optimality conditions . we recommended to extend these results for nonconvex function in normed spaces using the $\rho-$Housdoroff distance convergence.

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