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Collocation Spline method for solving Linear and Nonlinear Sixth-Order Boundary-Value Problems

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□ ABSTRACT □

In this paper, spline collocation method is considered for solving two forms of problems. The first form is general linear sixth-order boundary-value problem (BVP), and the second form is nonlinear sixth-order initial value problem (IVP). The existence, uniqueness, error estimation and convergence analysis of purpose methods are investigated. The study shows that proposed spline method with three collocation points can find the spline solutions and their derivatives up to sixth-order of the two BVP and IVP, thus is very effective tools in numerically solving such problems. Several examples are given to verify the reliability and efficiency of the proposed method. Comparisons are made to reconfirm the efficiency and accuracy of the suggested techniques.

Keywords: Sixth-Order Boundary Value Problems, Sixth-Order Initial Value Problems, Spline Functions, Collocation Points, Error Estimation, Convergence Analysis.

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طريقة الشريحة التجميعية لحل مسائل القيم الحدية الخطية وغير الخطية من المرتبة السادسة

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□ ملخّص □

في هذا العمل تم تقديم طريقة الشريحة التجميعية للحل العددي لنوعين من المسائل. النوع الأول هو مسألة القيمة الحدية في المعادلات التفاضلية الخطية المعممة من المرتبة السادسة و النوع الثاني هو مسألة القيمة الابتدائية في المعادلات التفاضلية غير الخطية المعممة من المرتبة السادسة. تم إثبات أن الطريقة المذكورة عند تطبيقها لمثل هذه المسائل تكون موجودة بشكل وحيد بالإضافة إلى تقدير الأخطاء وتحليل التقارب. تبين الدراسة أن طريقة الشريحة بثلاث نقاط تجميعية تستطيع إيجاد الحلول العددية الشرائحية ومشتقاتها حتى المرتبة السادسة للمسائل الخطية وغير الخطية المطروحة وبالتالي فهي أداة فعالة للحل العددي لمثل هذه المسائل. تم إثبات فعالية وكفاءة الطريقة المقترحة بحل عدد من مسائل الاختبار ومقارنة النتائج التي تم التوصل إليها مع نتائج لطرائق أخرى.

الكلمات المفتاحية: مسائل القيمة الحدية من المرتبة السادس، مسائل القيمة الابتدائية من المرتبة السادس، دوال شرائحية، نقاط مجمعة، تقدير الخطأ، تحليل التقارب.

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1. Introduction

Generally, sixth order boundary value problems arise in several branches of applied mathematics and physics.

In this paper, spline techniques with collocation points are presented for solving two forms of problems. The first form, general linear sixth-order BVP:

$$y^{(6)}(x)+\sum_{i=0}^{5}q_i(x)\ y^{(i)}(x)=g(x),\ \ x\in\ [a,b], \tag{1.1}$$

subject to the following two types of boundary conditions:

$$\text{Type I:}\quad \begin{cases} y(a)=\alpha_0,\ y'(a)=\alpha_1,\ y''(a)=\alpha_2, \\ y(b)=\beta_0,\ y'(b)=\beta_1,\ y''(b)=\beta_2 \end{cases}, \tag{1.2}$$

$$\text{Type II:}\quad \begin{cases} y(a)=\alpha_0,\ y''(a)=\alpha_2,\ y^{(4)}(a)=\alpha_4, \\ y(b)=\beta_0,\ y''(b)=\beta_2,\ y^{(4)}(b)=\beta_4, \end{cases}, \tag{1.3}$$

where α_i,β_i (i = 0, 1,2) and α_{2i},β_{2i} (i =2) are finite real constants and also g(x), $q_i(x)$ $(i=0,...,5)$ are all continuous functions on [a, b].

The second form, general nonlinear sixth-order initial value problem:

$$y^{(6)}(x)=f(x,y,y',y'',...,y^{(5)})\ ,\ x\in\ [a,b], \tag{1.4}$$

with the following initial conditions:

$$y^{(i)}(a)=\alpha_i,\ i=0,1,...,5\,. \tag{1.5}$$

The several spline methods have been extensively applied in numerical ordinary differential equations due to its easy implementation and high-order accuracy. Recently, various powerful numerical methods such as septic B-spline collocation method [1], quintic B-spline Collocation method [2], B-spline collocation method [3], septic spline collocation method [4], quintic non-polynomial spline method [5], and septic non-polynomial spline method [6] have been developed for obtaining approximate solutions to various types of sixth-order BVBs of the form (1.1) with one of boundary conditions (1.2)-(1.3). EL-Gamel at al. in [7] presented a Sinc-Galerkin method for finding numerical solutions of the form (1.1). Moreover, several other methods such as optimal homotopy asymptotic methods by Javed Ali at al. in [8, 9], differential quadrature method by Wang at al. in [10], a variation of parameters method by Mohyud-Din at al. in [11] and a differential transformation method by Ertürk in [12] have been developed for obtaining approximate analytic solutions to sixth-order BVBs.

2-Importance of Research and its Aim

Higher-order boundary value problems are known to arise in the study of astrophysics, hydrodynamic and hydro magnetic stability, fluid dynamics, astronomy, beam and long wave theory, engineering and applied physics. It is well known that the analysis solutions of those higher-order BVPs are either very difficult or not existent. For this cause, the numerical solutions of proposed problems are very important.

The work aims to develop spline collocation method of the numerical solutions for linear sixth-order boundary value problems, and nonlinear sixth-order initial value problems.

3-Methodology

Spline collocation techniques are presented for numerical spline solutions linear sixth-order BVPs (1.1)-(1.3) and nonlinear sixth-order IVP (1.4)-(1.5). In the beginning, each sixth-order BVP will be transformed into four IVPs. After that, spline approximations with three collocation points are applied directly into these IVPs for obtaining the spline solutions of BVPs (1.1) -(1.3). Moreover, spline techniques will be applied directly into nonlinear sixth-order IVP (1.4)-(1.5) without its reduction to system of first-order differential equations

3.1 Solution Scheme of sixth-order BVPs

Consider the sixth-order BVP:

$$y^{(6)}(x) = -\sum_{i=0}^{5} q_i(x)\ y^{(i)}(x) + g(x), \quad x \in [a, b], \tag{3.1}$$

subject to the following two types I and II of boundary conditions (1.2)-(1.3).

Let $y(x)$ be the unique solution to the BVP (3.1) with conditions of type I or type II. Then this solution is associated by a linear combination consist of four special sixth-order IVPs for this BVP. To find it, we assume that $U_0(x)$ is the unique solution to the following first IVP:

$$U_0^{(6)}(x) = -\sum_{i=0}^{5} q_i(x) U_0^{(i)}(x) + g(x), \quad a \le x \le b \ , \tag{3.2}$$

with the following initial conditions:

Type I: $U_0^{(i)}(a) = \alpha_i, \ (i = 0,1,2), \ U_0^{(i)}(a) = 0, \ (i = 3,4,5),$ (3.2a)

Type II: $U_0^{(2i)}(a) = \alpha_i, \ U_0^{(2i+1)}(a) = 0, \ (i = 0,1,2)$. (3.2b)

In addition, suppose that $U_1(x)$ is the unique solution to the second homogeneous IVP:

$$U_1^{(6)} = -\sum_{i=0}^{5} q_i(x) U_1^{(i)}(x), \quad a \le x \le b\,, \tag{3.3}$$

with the following initial conditions:

Type I: $U_1^{(i)}(a) = 0, \ (i = 0,1,2), \ U_1^{(3)}(a) = 1, \ U_1^{(i)}(a) = 0 \ \ (i = 4,5),$ (3.3a)

Type II: $U_1^{(2i)}(a) = 0\,(i = 0,1,2), \ U_1'(a) = 1, U_1^{(2i+1)}(a) = 0, \ (i = 1,2)$. (3.3b)

Let $U_2(x)$ be the unique solution to the following third IVP:

$$U_2^{(6)} = -\sum_{i=0}^{5} q_i(x) U_2^{(i)}(x), \quad a \le x \le b \ , \tag{3.4}$$

with the following initial conditions:

Type I: $U_2^{(i)}(a) = 0, \ (i = 0,1,...,3), \ U_2^{(4)}(a) = 1, \ U_2^{(5)}(a) = 0,$ (3.4a)

Type II: $U_2^{(2i)}(a) = 0\,(i = 0,1,2), \ U_2'(a) = 0, U_2^{(3)}(a) = 1, U_2^{(5)}(a) = 0,$ (3.4b)

Let $U_3(x)$ be the unique solution to the following fourth IVP:

$$U_3^{(6)} = -\sum_{i=0}^{5} q_i(x) U_3^{(i)}(x), \quad a \le x \le b\,, \tag{3.5}$$

Types I, II: $U_3^{(i)}(a) = 0, \ (i = 0,1,...,4), \ U_3^{(5)}(a) = 1$. (3.5a)

Moreover, there exist the linear combinations for three real constants $\hat{c}_1, \hat{c}_2$ and $\hat{c}_3$, such that:

$$y(x) = U_0(x) + \sum_{j=1}^{3} \hat{c}_j U_j \ , \tag{3.6}$$

is a unique solution to the sixth-order BVP (3.1) corresponding to the type (1.2) or type (1.3), as seen by the following computations:

$$y^{(6)}(x) = U_0^{(6)}(x) + \sum_{j=1}^{3} \hat{c}_j U_j^{(6)} =$$

$$=-\sum_{i=0}^{5} q_i(x)U_0^{(i)}(x)+g(x)+\sum_{j=1}^{3}\hat{c}_j[-\sum_{i=0}^{5} q_i(x)U_j^{(i)}(x)]$$

$$=-\sum_{i=0}^{5} q_i(x)[U_0^{(i)}(x)+\sum_{j=1}^{3}\hat{c}_j U_j^{(i)}(x)]+g(x) \; =-\sum_{i=0}^{5} q_i(x)\; y^{(i)}(x)+g(x),$$

where $\quad y^{(i)}(x)=U_0^{(i)}(x)+\sum_{j=1}^{3}\hat{c}_j U_j^{(i)}, i=0,1,...,5$.

Now, it will be illustrated that the relation (3.6) which it is the solution $y(x)$ to the BVP (3.1), (1.2) holds on the boundary conditions (1.2), thus satisfying this conditions (1.2) yield out:

$$y^{(i)}(a)=U_0^{(i)}(a)+\sum_{j=1}^{3} c_j\, U_j^{(i)}(a)=\alpha_i+\sum_{j=1}^{3} c_j(0)=\alpha_i\,, \quad (i=0,1,2),$$

where $\hat{c}_j=c_j$ (j=1,2,3).

The unknown constants c_1, c_2, and c_3 will be determined from the remainder conditions of

type (1.2) by solving the following 3×3 system of linear equations:

$$y^{(i)}(b)=U_0^{(i)}(b)+\sum_{j=1}^{3} c_j\, U_j^{(i)}(b)\equiv\beta_i\,, \quad (i=0,1,2)\,. \tag{3.7}$$

Moreover, if the solution $y(x)$ is corresponded to the BVP (3.1),(1.3), then satisfying boundary conditions of the type (1.3), we get:

$$y^{(2i)}(a)=U_0^{(2i)}(a)+\sum_{j=1}^{3}\hat{c}_j\, U_j^{(2i)}(a)=\alpha_{2i}+\sum_{j=1}^{3} c_j(0)=\alpha_{2i}\,, \;\; \hat{c}_j=\bar{c}_j \;\; (i=0,1,2)\;.$$

The unknown constants $\bar{c}_1, \bar{c}_2$, and $\bar{c}_3$ will are determined from the remainder end boundary conditions of type (1.3), namely, the 3×3 system of linear equations:

$$y^{(2i)}(b)=U_0^{(2i)}(b)+\sum_{j=1}^{3}\bar{c}_j\, U_j^{(2i)}(b)\equiv\beta_{2i}\,, \quad (i=0,1,2)\,. \tag{3.8}$$

It is apparent that for solving the BVP (3.1),(1.2), we necessitate to solve four IVPs (3.2)-(3.5) with conditions (3.2a)-(3.5a) of the type I. Similarly, for the BVP (3.1), (1.3), we need to solve the IVPs (3.2)-(3.5), with conditions (3.2b)-(3.4.b)-(3.5a) of the type II.

Now, spline techniques with three collocation points will be applied for sixth-order IVPs.

3.2 Formulation of the Spline Approximations.

Consider the uniform grid partition $\Delta:\equiv\{a=x_0<x_1<\cdots<x_k<x_{k+1}<\cdots<x_N=b\}$ of the interval $[a,b]$ with mesh size $h=(b-a)/N$ and grid points $x_k=a+k\;h,\;\; k=0,...,N$.

Let $S(x)\in C^6[a,b]$ be the spline approximation to the function $y(x)$ at the grid points is given:

$$S(x)=\sum_{i=0}^{6}\frac{(x-x_k)^i}{i!}S_k^{(i)}+\sum_{i=7}^{9}\frac{(x-x_k)^i}{i!}C_{k,i-6}\,, x\in[x_k,x_{k+1}],\; k=0,...,N-1 \;\;, \tag{3.9}$$

where $\quad S^{(i)}(a)=S_0^{(i)} \;\; (i=0,1,...,6)$.

By applying the spline approximation (3.9) and its derivatives up to sixth-order with respect to x, into sixth-order IVPs (3.2)-(3.5), with three collocation points

$x_{k+z_j} = x_k + h\, z_j$, ($j$=1,2,3), in each subinterval $\mathrm{I}_k = [x_k, x_{k+1}]$, k=0(1)N-1, then we have, respectively:

$$S_{U_0}^{(6)}(x_{k+z_j}) = -\sum_{i=0}^{5} g_i(x_{k+z_j}) S_{U_0}^{(i)}(x_{k+z_j}) + g(x_{k+z_j}) \;,\quad j = 1,2,3,\quad k = 0(1)N-1, \tag{3.10}$$

with the following initial conditions:

Type I: $S_{U_0}^{(i)}(a) = \alpha_i \;(i = 0,1,2),\;\; S_{U_0}^{(i)}(a) = 0 \;(i = 3,4,5)$, (3.10a)

Type II: $S_{U_0}^{(2i)}(a) = \alpha_{2i} \;\; S_{U_0}^{(2i+1)}(a) = 0 \;(i = 0,1,2)$. (3.10b)

$$S_{U_1}^{(6)}(x_{k+z_j}) = -\sum_{i=0}^{5} q_i(x_{k+z_j}) S_{U_1}^{(i)}(x_{k+z_j}) \;,\quad j = 1(1)3,\quad k = 0(1)N-1, \tag{3.11}$$

with the following initial conditions:

Type I: $S_{U_1}^{(i)}(a) = 0 \;(i = 0,1,2),\;\; S_{U_1}^{(3)}(a) = 1,\; S_{U_1}^{(i)}(a) = 0 \;\;(i = 4,5)$, (3.11a)

Type II: $S_{U_1}^{(2i)}(a) = 0 \;(i = 0,1,2),\;\; S_{U_1}^{(1)}(a) = 1,\; S_{U_1}^{(2i+1)}(a) = 0 \;\;(i = 1,2)$. (3.11b)

$$S_{U_2}^{(6)}(x_{k+z_j}) = -\sum_{i=0}^{5} q_i(x_{k+z_j}) S_{U_2}^{(i)}(x_{k+z_j}) \;,\quad j = 1(1)3,\quad k = 0(1)N-1, \tag{3.12}$$

with the following initial conditions:

Type I: $S_{U_2}^{(i)}(a) = 0 \;(i = 0,1,...,3),\;\; S_{U_2}^{(4)}(a) = 1,\; S_{U_2}^{(5)}(a) = 0,$ (3.12a)

Type II: $S_{U_2}^{(2i)}(a) = 0 \;(i = 0,1,2),\; S_{U_2}^{(1)}(a) = 0,\; S_{U_2}^{(3)}(a) = 1,\; S_{U_2}^{(5)}(a) = 0$. (3.12b)

$$S_{U_3}^{(6)}(x_{k+z_j}) = -\sum_{i=0}^{5} q_i(x_{k+z_j}) S_{U_3}^{(i)}(x_{k+z_j}) \;,\quad j = 1(1)3,\quad k = 0(1)N-1, \tag{3.13}$$

with the following initial conditions for two types I and II:

$S_{U_3}^{(i)}(a) = 0 \;(i = 0,1,...,4),\;\; S_{U_3}^{(5)}(a) = 1,$ (3.13a)

Now, by substituting spline solutions to the system of linear equations (3.7), the coefficients c_1, c_2, and c_3 the associated by conditions of type I , will be known as follow:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} S_{U_1}(b) & S_{U_2}(b) & S_{U_3}(b) \\ S'_{U_1}(b) & S'_{U_2}(b) & S'_{U_3}(b) \\ S''_{U_1}(b) & S''_{U_2}(b) & S''_{U_3}(b) \end{bmatrix}^{-1} \begin{bmatrix} \beta_0 - S_{U_0}(b) \\ \beta_1 - S'_{U_0}(b) \\ \beta_2 - S''_{U_0}(b) \end{bmatrix}. \tag{3.14}$$

Similarly, substituting spline solutions to the system of linear equations (3.8), the coefficients $\bar{c}_1, \bar{c}_2$, and $\bar{c}_3$ for conditions fitted of type II are obtained:

$$\begin{bmatrix} \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \end{bmatrix} = \begin{bmatrix} S_{U_1}(b) & S_{U_2}(b) & S_{U_3}(b) \\ S''_{U_1}(b) & S''_{U_2}(b) & S''_{U_3}(b) \\ S_{U_1}^{(4)}(b) & S_{U_2}^{(4)}(b) & S_{U_3}^{(4)}(b) \end{bmatrix}^{-1} \begin{bmatrix} \beta_0 - S_{U_0}(b) \\ \beta_2 - S''_{U_0}(b) \\ \beta_4 - S_{U_0}^{(4)}(b) \end{bmatrix}. \tag{3.15}$$

Hence, the spline solution and its derivatives $S^{(i)}(x), i = 0,1,...,6$ of the BVPs (3.1),(1.2)-(1.3) will be known from systems (3.14) and (3.15), namely:

$$S^{(i)}(x_k)=S^{(i)}_{U_0}(x_k)+\sum_{j=1}^{3}\hat{c}_j S^{(i)}_{U_j}(x_k),\ (i=0,1,...,6),\ (k=0,...,N),\qquad(3.16)$$

where $\hat{c}_j=c_j,\bar{c}_j,(j=1,2,3)$.

On the other hand, applying collocation points $x_{k+z_j}=x_k+h\,z_j$, (j=1,2,3) to (3.9), we obtain

$$S(x_{k+z_j})=\sum_{i=0}^{6}\frac{(h\,z_j)^i}{i!}S^{(i)}_k+\sum_{i=7}^{9}\frac{(h\,z_j)^i}{i!}C_{k,i-6},\quad j=1,2,3,\quad k=0,...,N-1.\qquad(3.17)$$

where $z_1=1/4$, $z_2=3/4$, $z_3=1$, $x_{k+z_j}\in]x_k,x_{k+1}]$, ($j$=1,2,3).

The first three coefficients $C_{k,1},C_{k,2},C_{k,3}$ will be computed in each iteration from the equations system (3.17) by using the initial value conditions if k=0, or from the previous steps if $k\geq 1$.

3.3 Spline Solution of Nonlinear Sixth-Order IVP

The numerical spline solution of nonlinear sixth-order IVP (1.4)-(1.5) by presented spline method is more easy than BVPs, because the spline approximation (3.9) and its derivatives $S^{(i)}(x), i=0,...,6$, will be applied directly into nonlinear sixth-order IVP without its reduction to system of first-order differential equations. Now, spline collocation method is applied into (1.4)-(1.5), with collocation points $x_{k+z_j}=x_k+h\,z_j$, (j=1,2,3), in subinterval $\mathrm{I}_k=[x_k,\,x_{k+1}]$,

k=0(1)N-1, as follow:

$$S^{(6)}(x_{k+z_j})=f[x_{k+z_j},S(x_{k+z_j}),S'(x_{k+z_j}),...,S^{(5)}(x_{k+z_j})]\ ,\ x_k\in\ [a,b],$$

with the following initial conditions:

$$S^{(i)}_0(a)=\alpha_i,\ i=0,1,...,5\,.$$

where $z_1=1/4$, $z_2=3/4$, $z_3=1$, $x_{k+z_j}\in]x_k,x_{k+1}]$, ($j$=1,2,3).

3.4 A unique Spline Collocation Solution

Consider the following sixth-order IVP:

$$\begin{cases} y^{(6)}(x)=F[x,y(x),y'(x),...,y^{(5)}(x)],\ \ x\in[a,b] \\ y^{(d)}(a)=a_d,\ \ d=0(1)5. \end{cases}\qquad(3.18)$$

Suppose that $F:[a,\,b]\times C[a,\,b]\times\cdots\times C^5[a,\,b]\to R$ is an enough smooth function satisfying the following Lipschitz condition in respect to the last argument:

$$|\,F(x,y_0,...,y_5)-F(x,\ddot{y}_0,...,\ddot{y}_5)\,|\leq L\sum_{i=0}^{5}|\,y_i-\ddot{y}_i\,|,$$

$$\forall\ (x,y_0,...,y_5),(x,\ddot{y}_0,...,\ddot{y}_5)\in[a,\,b]\times R^6$$

where the constant L is called a Lipschitz constant for F.

By applying the Spline approximation (3.9) and its derivatives into the problem (3.18),

with three collocation points $x_{k+z_j}=x_k+z_j\,h$, (j=1,2,3), we obtain the linear system:

$$S_k^{(6)} + (h z_j) C_{k,1} + \frac{(h z_j)^2}{2} C_{k,2} + \frac{(h z_j)^3}{3!} C_{k,3} = F[x_{k+z_j},\ S(x_{k+z_j}),\dots, S^{(5)}(x_{k+z_j})],$$
$$j = 1,2,3, \quad k = 0,\dots, N-1, \qquad (3.19)$$

$$S^{(d)}(a) = a_d,\ d = 0(1)5. \qquad (3.20)$$

We rewrite (3.19) in the matrices formula:

$$A\,\overline{C}_k = \hat{F}_k - \hat{S}_k \qquad (3.21)$$

where

$$A = \begin{bmatrix} h z_1 & \frac{h^2 z_1^2}{2!} & \frac{h^3 z_1^3}{3!} \\ h z_2 & \frac{h^2 z_2^2}{2!} & \frac{h^3 z_2^3}{3!} \\ h & \frac{h^2}{2!} & \frac{h^3}{3!} \end{bmatrix},\quad \overline{C}_k = \begin{bmatrix} C_{k,1} \\ C_{k,2} \\ C_{k,3} \end{bmatrix},\quad \hat{F}_k = \begin{bmatrix} F_{k+z_1} \\ F_{k+z_2} \\ F_{k+1} \end{bmatrix},\ \hat{S}_k = \begin{bmatrix} S_k^{(6)} \\ S_k^{(6)} \\ S_k^{(6)} \end{bmatrix},$$

$$F_{k+z_j} = F[x_{k+z_j},\ S(x_{k+z_j}),\dots, S^{(5)}(x_{k+z_j})], \quad j=1,2,3.$$

Theorem : Suppose that $F:[a,\ b]\times C[a,\ b]\times\cdots\times C^5[a,\ b] \to R$ satisfies Lipschitz condition, and if

$$h^4 < \frac{1290240}{4351\,L} \qquad (3.22)$$

then the spline solution S(x) exists and is uniquely defined when $z_1 = 1/4$, $z_2 = 3/4$, $z_3 = 1$, where L is a Lipschitz constant of F.

Proof. It is sufficient to prove that the vector $\overline{C}_k$ can be uniquely determined for arbitrary given $\overline{S}_k$. We note that if $z_1 = 1/4$, $z_2 = 3/4$, $z_3 = 1$, then the matrix A^{-1} exists and is

nonsingular because $Det(A) = \frac{3h^6}{2048}$.

Let $\overline{C}_k^1, \overline{C}_k^2 \in R^3$, then using $\|\,.\,\|_1$ from (3.21), we have

$$\overline{C}_k^1 = A^{-1}\hat{F}_k^1 - A^{-1}\hat{S}_k \ \text{ and } \ \overline{C}_k^2 = A^{-1}\hat{F}_k^2 - A^{-1}\hat{S}_k$$

Thus $\overline{C}_k^1$ and $\overline{C}_k^2$ can be written in the form

$$\overline{C}_k^1 = \overline{Q}_k(C_{k,1}^1, C_{k,2}^1, C_{k,3}^1,\ h) \ \text{ and } \ \overline{C}_k^2 = \overline{Q}_k(C_{k,1}^2, C_{k,2}^2, C_{k,3}^2,\ h)$$

Applying $\|\,.\,\|_1$, Lipschitz condition and using *Mathematica*, we get

$$\|\overline{Q}_k(\overline{C}_k^1) - \overline{Q}_k(\overline{C}_k^2)\| = \|(A^{-1}\hat{F}_k^1) - (A^{-1}\hat{F}_k^2)\| = \|A^{-1}(\hat{F}_k^1 - \hat{F}_k^2)\| \le$$
$$\{L_1 H_1 |C_{k,1}^1 - C_{k,1}^2| + L_2 H_2 |C_{k,2}^1 - C_{k,2}^2| + L_3 H_3 |C_{k,3}^1 - C_{k,3}^2|\} \le$$
$$\{L_1(\frac{4351}{1290240}h^4)|C_{k,1}^1 - C_{k,1}^2| + L_2(\frac{75077}{165150720}h^5)|C_{k,2}^1 - C_{k,2}^2| +$$
$$L_3(\frac{3299}{61931520}h^6)|C_{k,3}^1 - C_{k,3}^2|\}$$
$$\le L(\frac{4351}{1290240}h^4)\{|C_{k,1}^1 - C_{k,1}^2| + |C_{k,2}^1 - C_{k,2}^2| + |C_{k,3}^1 - C_{k,3}^2|\},$$

where

$L = \max(L_1, L_2, L_3)$,

$H_1 = \frac{143}{30720}h^4 - \frac{13}{9216}h^5 + \frac{11}{86016}h^6 \le \frac{4351}{1290240}h^4$, $\forall\ h \in]0,1[$,

$H_2 = \frac{13}{20480}h^5 - \frac{16379}{82575360}h^6 + \frac{143}{7864320}h^7 \le \frac{75077}{165150720}h^5$, $\forall\ h \in]0,1[$,

$H_3 = \frac{3713}{49545216}h^6 - \frac{169}{7077888}h^7 + \frac{13}{5898240}h^8 \le \frac{3299}{61931520}h^6$, $\forall\ h \in]0,1[$,

$\max(H_1, H_2, H_3) = H_1 \le \frac{4351}{1290240}h^4$, $\forall\ h \in]0,1[$,

$$A^{-1} = \begin{bmatrix} \frac{8}{h} & -\frac{8}{3h} & \frac{1}{h} \\ -\frac{112}{3h^2} & \frac{80}{3h^2} & -\frac{32}{3h^2} \\ \frac{64}{h^3} & -\frac{64}{h^3} & \frac{32}{h^3} \end{bmatrix}. \tag{3.23}$$

Thus, the function $\overline{Q}_k$ defines a contraction mapping if $h^4 L \frac{4351}{1290240} < 1$ which satisfies (3.22). Hence there exists a unique $\overline{C}_k$ that satisfies $\overline{C}_k = \overline{Q}_k(C_{k,1}, C_{k,2}, C_{k,3}, C_{k,4}, h)$ which may be found by iterations, $\overline{C}_k^{p+1} = \overline{Q}_k(\overline{C}_k^{p}, , h)$, p=0,1,2,… and this completes the proof.

3.5 Error Estimation and Convergence Analysis

We assume that $y(x) \in C^{10}[a, b]$, the unique solution of the sixth-order IVP and $S(x)$ be a spline approximation solution to $y(x)$, also $T = (\overline{\tau}_k)$ is a 3-dimensional column vector. Here, the vector $\overline{\tau}_k$ is the local truncation error. Applying the Spline solution $S(x)$ on three collocation points $x_{k+z_j} = x_k + z_j h$, (j=1,2,3), putting $y(x_{k+z_j}) = y(x_k + hz_j)$, $S_k^{(m)} = S^{(m)}(x_k)$ and $y_k^{(m)} = y^{(m)}(x_k)$, (m=0,…,5), k=0,…,N-1, for $z_1 = 1/4$, $z_2 = 3/4$, $z_3 = 1$, we obtain the local truncation error formula:

$$\overline{\tau}_k = \mathrm{M}\,\overline{C}_k + \overline{\Psi}_k, \tag{3.24}$$

where

$$\overline{\Psi}_k = \begin{bmatrix} \sum_{i=0}^{6} \frac{(z_1 h)^i}{i!} S_k^{(i)} - y(x_k + z_1 h) \\ \sum_{i=0}^{6} \frac{(z_2 h)^i}{i!} S_k^{(i)} - y(x_k + z_2 h) \\ \sum_{i=0}^{6} \frac{h^i}{i!} S_k^{(i)} - y(x_k + h) \end{bmatrix}, \quad \mathrm{M} = \begin{bmatrix} \frac{(z_1 h)^7}{7!} & \frac{(z_1 h)^8}{8!} & \frac{(z_1 h)^9}{9!} \\ \frac{(z_2 h)^7}{7!} & \frac{(z_2 h)^8}{8!} & \frac{(z_2 h)^9}{9!} \\ \frac{h^7}{7!} & \frac{h^8}{8!} & \frac{h^9}{9!} \end{bmatrix},$$

$$\overline{C}_k = \begin{bmatrix} C_{k,1} \\ C_{k,2} \\ C_{k,3} \end{bmatrix}$$

On the other hand, from the system (3.21), we get

$$\overline{C}_k = A^{-1} \hat{F}_k - A^{-1} \hat{S}_k \tag{3.25}$$

where A^{-1} is the matrix (3.23), and $\hat{F}_k = [y^{(6)}(x_{k+z_1}) \quad , y^{(6)}(x_{k+z_2}) \quad , y^{(6)}(x_{k+1})]^T$.

Using Taylor's expansions for the functions $y^{(m)}(x), m = 0,...,5$ about x_k, in the relation (3.25) and substituting into (3.24), we get the local truncation error at the *k*th step as follows:

$$\overline{\tau}_k = \mathrm{M}(A^{-1} \hat{F}_k + A^{-1} \hat{S}_k) + \overline{\Psi}_k = \begin{bmatrix} \frac{473}{2097152*10!} h^{10} y^{(10)}(x_k) \\ \frac{41553}{20971520*8!} h^{10} y^{(10)}(x_k) \\ \frac{23}{32*10!} h^{10} y^{(10)}(x_k) \end{bmatrix} \quad ,k=0,1,..,N \tag{3.26}$$

where

$$y(x) = \sum_{i=0}^{9} \frac{(x - x_k)^i}{i!} y^{(i)}(x_k) + O(h^{10}), \; x \in [x_k, x_{k+1}] ,$$

Note from the relation (3.26) that the proposed Spline collocation method is exact for expansions of the solution of degree ≤ 9, hence we have $\| T \|_\infty = O(h^9)$.

Consequently, we have obtained the following: let $y \in C^9[a, b]$ be Lipschitz continuous, then the spline approximation $S(x)$ converges to the solution $y(x)$ of the sixth-order BVP as $h \to 0$ for $z_1 = 1/4, z_2 = 3/4, z_3 = 1$ and

$$\lim_{h \to 0} S^{(m)}(e) = y^{(m)}(e), \; m = 0,...,6, \quad e = a, b \; .$$

Furthermore, we have:

$$| y^{(m)}(x) - S^{(m)}(x) | < C_m h^{9-m}, \; m = 0,...,5 \; .$$

4. Numerical Results and Discussion

The experiments below are designed to test the efficiency of the spline methods when applied to five linear sixth-order BVPs and also when applied to two nonlinear sixth-order IVPs. These problems have exact solutions, thus we compute their actual errors. In calculations, the notations $\delta^{(k)} = \max \| y^{(k)}(x) - S^{(k)}(x) \|$ are used to denote maximum absolute errors, where, topgallant notation k indicate orders of derivatives. Here, the numerical results for all problems are computed by using **Turbo Pascal** programs under Windows (***TPW* 1.5**) in double precision.

Problem 1. Consider the following sixth-order BVP (cf. [5, 6]):

$$y^{(6)} + y = 6[2x\cos(x) + 5\sin(x)],\ -1 \le x \le 1, \tag{4.1}$$

$$\text{Type I}:\begin{cases} y(-1)=0, & y'(-1)=2\sin(1), \quad y''(-1)=-4\cos(1)-2\sin(1), \\ y(1)=0, & y'(1)=2\sin(1), \quad y''(1)=4\cos(1)+2\sin(1) \end{cases} \tag{4.2}$$

$$\text{Type II}:\begin{cases} y(0)=0, & y''(0)=0, \quad y^{(4)}(0)=0, \\ y(1)=0, & y''(1)=4\cos(1)+2\sin(1), \quad y^{(4)}(1)=-12\sin(1)-8\cos(1) \end{cases} \tag{4.3}$$

The exact solution is $y(x)=(x^2-1)\sin(x)$. **Table 1** shows spline solutions of *BVP* (4.1)-(4.2) for *N, 2N, 4N, N*=10 by presented method. In **Table 2**, the rate of convergence is computed when the proposed method applied to BVP (4.1)-(4.2). The nodal difference error ε_k^N, is defined by $\varepsilon_k^N = |S_k^N - S_{2k}^{2N}|$, k=1,…,N, where S_k^N is the spline solution at x_k by proposed method. The experimental nodal rate of convergence is given by $Rate = Log_2(\varepsilon_k^N / \varepsilon_{2k}^{2N})$.

Table 3 shows the maximum absolute errors of BVP (4.1),(4.3) by presented spline method are compared with sixth-order spline method [**5**] and fourth-order spline method [**6**].

Figures 1-2, depict the spline solution $y(x)$ with the exact solution and the absolute error for BVP (4.1), (4.3) obtained by proposed Spline method, respectively.

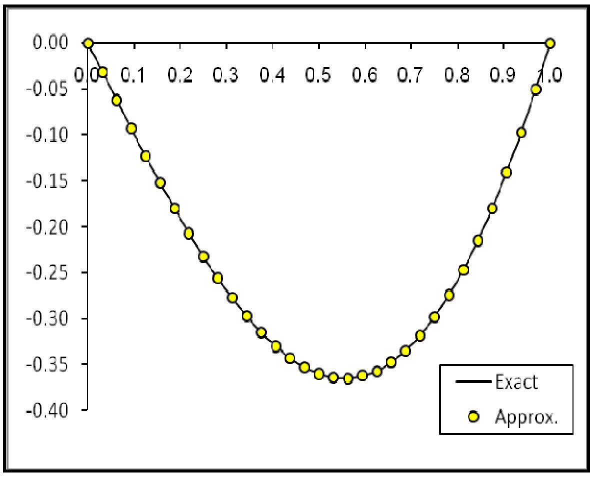

Fig.1: The spline solution S(x) and the exact solution y(x), for N=32.

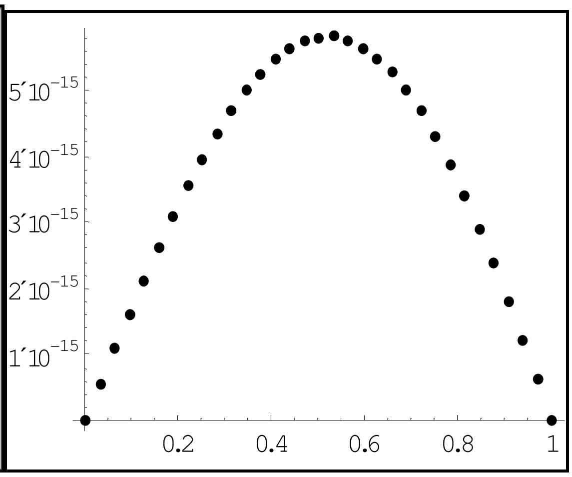

Fig.2: The absolute error of spline solution by the Spline method, for N=32.

Table 1: *spline solutions of BVP* (4.1)-(4.2) *by presented spline method for N*=10

k	x_k	S_k^N	S_{2k}^{2N}	S_{4k}^{4N}
1	-0.8	0.258248192724136794	0.258248192723824083	0.258248192723828094
2	-0.6	0.361371182972421415	0.361371182972799508	0.361371182972822182
3	-0.4	0.327111407538449438	0.327111407539228778	0.327111407539265726
4	-0.2	0.190722557562447868	0.190722557563229819	0.190722557563258250
5	0.0	0.000000000000000033	0.000000000000000001	-0.000000000000000001
6	0.2	-0.190722557562447864	-0.190722557563229816	-0.190722557563258251

7	0.4	-0.327111407538449433	-0.327111407539228774	-0.327111407539265727
8	0.6	-0.361371182972421410	-0.361371182972799502	-0.361371182972822182
9	0.8	-0.258248192724136787	-0.258248192723824075	-0.258248192723828093

Table 2: ***The rate of convergence for presented spline method, with N*=10.**

k	ε_k^N	ε_{2k}^{2N}	Rate of convergence
1	3.1271E-13	4.01E-15	6.285
2	3.7809E-13	2.267E-14	4.060
3	7.7934E-13	3.695E-14	4.3987
4	7.8195E-13	2.843E-14	4.782
5	3.200E-17	2.000 E-18	4.04439
6	7.8195E-13	2.844E-14	4.781
7	7.7934E-13	3.695E-14	4.3985
8	3.7809E-13	2.268E-14	4.059
9	3.1271E-13	4.02E-15	6.282

Table 3: ***The maximum absolute errors of BVP* (4.1),(4.3).**

h	Fourth-order Spline method [6]	Sixth-order Spline method [5]	Presented spline method
1/8	1.652489367 E-08	9.4483046941E-09	2.383422 E-11
1/16	2.497231310 E-10	1.509762315 E-10	3.713706 E-13
1/32	2.125805087 E-11	2.370159624 E-12	5.802460 E-15
1/64	--------------	4.596323320 E-14	9.069351 E-17

Problem 2. Consider the following sixth-order BVP (cf. [1,11]):

$$y^{(6)} - y = -6\ \exp(x), \qquad 0 \le x \le 1, \tag{4.4}$$

$$\text{Type I}: \ y(0)=1\ ,\ y'(0)=0,\ y''(0)=-1,\ y(1)=0\ ,\ y'(1)=-e,\ y''(1)=-2e\ , \tag{4.5}$$

$$\text{Type II}: \ y(0)=1\ ,\ y''(0)=-1,\ y^{(4)}(0)=-3,\ y(1)=0\ ,\ y'(1)=-2e,\ y^{(4)}(1)=-4e\ . \tag{4.6}$$

The exact solution is $y(x) = (1-x)\,Exp(x)$. In Table 4, the absolute errors of BVPs (4.4)-(4.6), with h=0.1 are given by using presented spline method, septic B-spline collocation method [1] and variation of parameters method [11].

Table 4: ***The absolute errors of BVPs* (4.4)-(4.6).**

x_i	Spline Method [1] for Type I	Variation of Parameters Method [11] for Type II	Presented Method for Type II	Presented Method for Type I	Exact Solution	Spline Solution for Type II
0.1	1.215935E-05	7.066E - 12	6.5952E-13	2.4519E-16	0.9946538262681	0.9946538262674
0.2	2.741814E-05	1.451E - 11	1.2548E-12	1.1413E-15	0.9771222065281	0.9771222065269
0.3	2.205372E-06	2.265E - 11	1.7314E-12	2.4672E-15	0.9449011653032	0.9449011653015
0.4	5.483627E-06	3.164E - 11	2.0465E-12	3.6934E-15	0.8950948185848	0.8950948185827
0.5	2.503395E-06	4.147E - 11	2.1702E-12	4.3088E-15	0.8243606353501	0.8243606353479
0.6	1.621246E-05	5.189E - 10	2.0874E-12	4.0451E-15	0.7288475201562	0.7288475201541
0.7	2.068281E-05	6.241E - 10	1.7994E-12	2.9896E-15	0.6041258122411	0.6041258122393

0.8	2.261996E-05	7.229E - 11	1.3254E-12	1.5717E-15	0.4451081856985	0.4451081856972
0.9	1.946092E-05	8.053E - 13	7.0508E-13	4.1003E-16	0.2459603111157	0.2459603111150
1	---------	8.562E - 11	2.3717E-20	5.9631E-19	0.0000000000000	0.0000000000000

Problem 3. Consider the following stiff BVP [3]:

$$\begin{cases} y^{(6)} - (1+c)y^{(4)} + c\,y'' - c\,y = 0 \quad , \qquad 0 \le x \le 1, \\ y(0) = y'(0) = 1,\ y''(0) = 0,\ y(1) = \frac{7}{6} + \sinh(1),\ y'(1) = \frac{1}{2} + \cosh(1),\ y''(1) = 1 + \sinh(1). \end{cases}$$

The exact solution is $y(x) = 1 + \frac{1}{6}x^2 + \sinh(x)$. We solve this problem for c=1 with N = 8, 16,…64. **Table** 5 appears the maximum absolute errors for problem3 are given by using the presented method and the B-Spline collocation method [3]. In **Table6**, the maximum absolute errors $\delta^{(i)}, (i = 0,...,6)$ are computed by presented spline method for N=10, and c=100.

Table 5*: The maximum absolute errors for problem 3*, *with* c=1.

h	Sixth-order B-Spline collocation method [3]				Presented spline method			
	$\delta^{(0)}$	$\delta^{(1)}$	$\delta^{(2)}$	$\delta^{(3)}$	$\delta^{(0)}$	$\delta^{(1)}$	$\delta^{(2)}$	$\delta^{(3)}$
1/8	2.39E−11	1.87E−10	1.011E−9	3.88E−7	5.19 E-16	1.83 E-15	2.18 E-14	1.6 E-13
1/16	3.75E−13	2.94E−12	1.59E−11	2.47E−8	5.00E-18	1.66 E-17	1.42 E-16	9.80 E-16
1/32	5.87E−15	4.61E−14	2.49E−13	1.55E−9	6.51 E-19	1.52 E-18	4.66 E-18	5.51 E-17
1/64	9.18E−17	7.21E−16	3.89E−15	9.7E−11	7.59 E-19	1.84 E-18	9.43 E-18	7.37 E-17

Table 6: *Maximum abs errors for problem 3*, *with* c=100 , *h*=0.1.

$\delta^{(0)}$	$\delta^{(1)}$	$\delta^{(2)}$	$\delta^{(3)}$	$\delta^{(4)}$	$\delta^{(5)}$	$\delta^{(6)}$
2.91E-16	9.88E-16	9.76E-15	2.55E-13	1.57E-11	5.90E-11	1.59E-09

Problem 4. Consider the following BVP [4]:

$$\begin{cases} y^{(6)} + y^{(3)} + y^{(2)} - y = e^{-x}(-15x^2 + 78x - 114),\ 0 \le x \le 1, \\ y(0) = y'(0) = y''(0) = 0,\ y(1) = 1/e,\ \ y'(1) = 2/e\ ,\ \ y''(1) = 3/e. \end{cases}$$

The exact solution is $y(x) = x^3 e^{-x}$. In **Table7**, the maximum absolute errors $\delta^{(i)}(x)$ (i=0,…,5) are calculated by proposed spline method, and comparison with other spline method [4]. The spline solutions $S^{(i)}(x), (i = 0,...,5)$ as well as the exact solutions $y^{(i)}(x), (i = 0,...,5)$ are illustrated in **Figs. 3-8,** respectively.

Table 7: *The maximum absolute errors for problem 4.*

h	Spline collocation method [4]	Presented spline method					
	$\delta^{(0)}$	$\delta^{(0)}$	$\delta^{(1)}$	$\delta^{(2)}$	$\delta^{(3)}$	$\delta^{(4)}$	$\delta^{(5)}$
1/8	2.3028E−006	6.09 E-13	2.08 E-12	1.87 E-11	1.52 E-10	2.24 E-08	1.30 E-08
1/16	5.6948E−007	6.90E-15	2.36 E-14	1.81 E-13	2.18 E-12	3.70 E-10	1.98 E-10
1/32	1.4237E −007	9.73 E-19	3.36 E-16	2.40 E-15	3.32 E-14	5.96 E-12	3.07 E-12

1/64	3.5646E –008	1.47 E-18	6.29 E-18	3.63 E-17	5.58 E-16	9.46 E-14	4.80 E-14
1/128	8.8478E –009	1.00 E-18	1.88 E-18	1.02 E-17	1.12 E-16	1.97 E-15	1.77 E-15

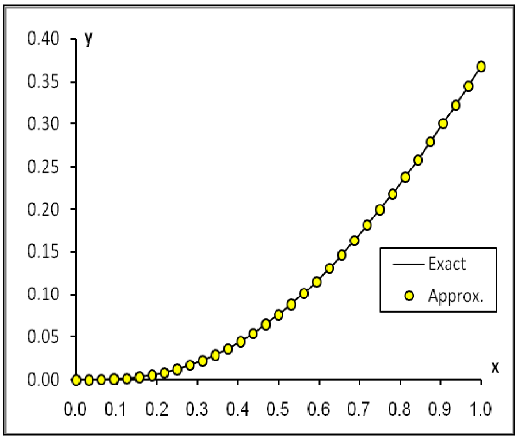

***Fig*.3**: *The spline solution S(x) and the exact solution y(x), for Problem4, N*=32.

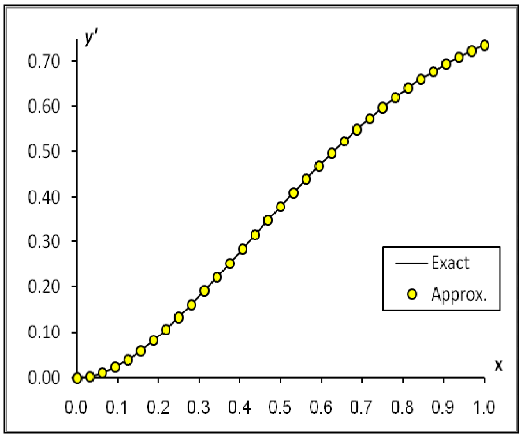

***Fig*.4**: *The spline solution* S'(*x*) *and the exact solution y'*(*x*)*, for Problem4, N*=32.

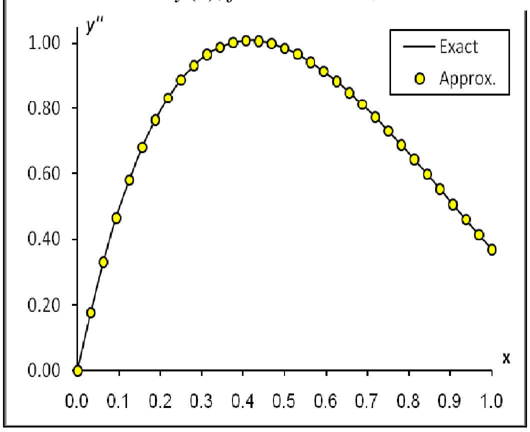

***Fig*.5**: *The spline solution S''*(*x*) *and the exact solution y''*(*x*)*, for Problem4, N*=32.

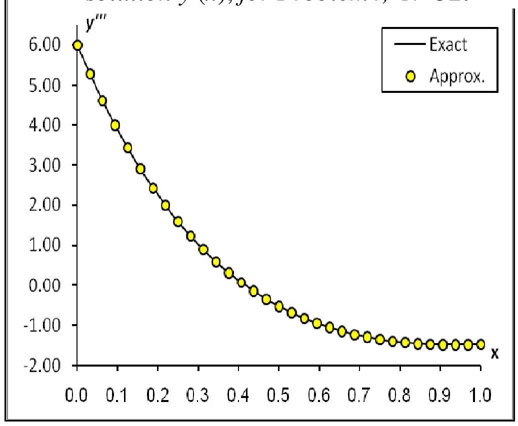

***Fig*.6**: *The spline solution S'''*(*x*) *and the exact solution y'''*(*x*)*, for Problem4, N*=32.

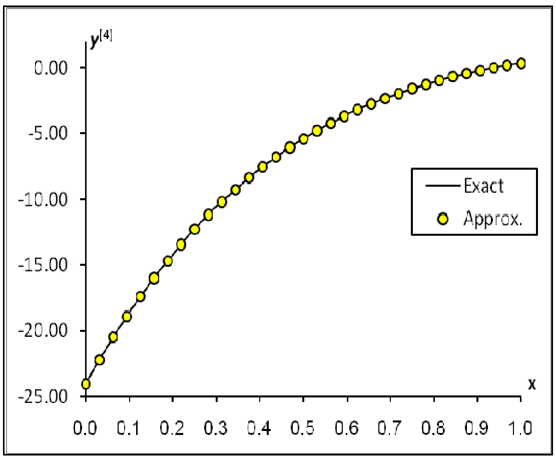

***Fig*.7**: *The spline solution* $S^{(4)}(x)$ *and the exact solution* $y^{(4)}(x)$*, for Problem4, N*=32.

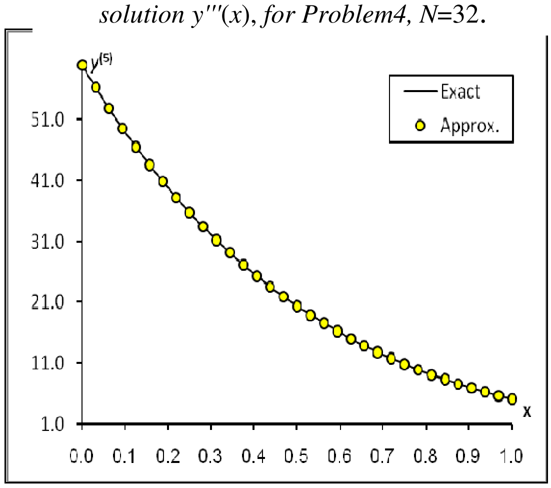

***Fig*.8**: *The spline solution* $S^{(5)}(x)$ *and the exact solution* $y^{(5)}(x)$*, for Problem4, N*=32.

Problem 5. Consider the following BVP (cf. [1, 2]):

$$\begin{cases} y^{(6)} + x\,y = -(24+11x+x^3)e^x, \quad 0 \le x \le 1, \\ y(0)=0,\ y'(0)=1,\ y''(0)=0,\ y(1)=0,\ y'(1)=-e\ , y''(1)=-4e. \end{cases}$$

Its exact solution is $y(x) = x(1-x)\exp(x)$. **Table** 8 shows the absolute errors for problem 5 are obtained by presented spline method, spline collocation method [1] and quintic b-spline collocation method [2].

Table 8: *The absolute errors for problem 5, with h=0.1.*

x_i	Spline collocation method[1]	Quintic B-spline Collocation Method [2]	Presented Spline method	Exact Solution	Spline Solution
0.1	5.014241E-06	4.619360E-07	2.524151E-15	0.09946538262681	0.09946538262681
0.2	1.330674E-05	1.847744E-06	1.212672E-14	0.19542444130563	0.19542444130564
0.3	1.162291E-06	3.874302E-06	2.655593E-14	0.28347034959096	0.28347034959099
0.4	1.490116E-07	6.288290E-06	3.999286E-14	0.35803792743390	0.35803792743394
0.5	8.106232E-06	7.361174E-06	4.675229E-14	0.41218031767503	0.41218031767508
0.6	2.160668E-05	7.241964E-06	4.384413E-14	0.43730851209372	0.43730851209377
0.7	2.643466E-05	6.496906E-06	3.225594E-14	0.42288806856880	0.42288806856883
0.8	2.858043E-05	4.649162E-06	1.679123E-14	0.35608654855879	0.35608654855881
0.9	2.457201E-05	2.324581E-06	4.286624E-15	0.22136428000413	0.22136428000413

Problem 6. We consider the following nonlinear IVP :

$$\begin{cases} y^{(6)} = \exp(-x)\, y^2(x)\ , \qquad 0 \le x \le 1, \\ y(0) = y'(0) = y''(0) = \cdots = y^{(5)}(0) = 1. \end{cases}$$

The exact solution is $y(x) = \exp(x)$. **Table 9** appears comparisons of the numerical solution and absolute errors by presented spline method In **Table 10**, maximum absolute errors are computed by presented spline method for N=10.

Table 9: The numerical solution and abs errors of problem 6, for *h*=0.1.

x_i	Presented spline methods		
	Exact Solution	Spline Solution	Abs Error
0.1	1.1051709180756	1.1051709180756	2.86229373536173E-17
0.2	1.2214027581602	1.2214027581602	6.30029882431327E-16
0.3	1.3498588075760	1.3498588075760	3.37783186837859E-15
0.4	1.4918246976413	1.4918246976413	1.09723428259878E-14
0.5	1.6487212707001	1.6487212707002	2.73197263422897E-14
0.6	1.8221188003905	1.8221188003906	5.76176476321799E-14
0.7	2.0137527074705	2.0137527074706	1.08454261196766E-13
0.8	2.2255409284925	2.2255409284927	1.87916305779967E-13
0.9	2.4596031111569	2.4596031111573	3.05709667650089E-13
1.0	2.7182818284590	2.7182818284595	4.73292629046829E-13

Table 10: Maximum abs errors of Problem 6, for *h*=0.1.

$\delta^{(0)}$	$\delta^{(1)}$	$\delta^{(2)}$	$\delta^{(3)}$	$\delta^{(4)}$	$\delta^{(5)}$	$\delta^{(6)}$
4.73E-13	1.97E-12	6.31E-12	1.43E-11	6.31E-12	1.32E-12	1.17E-11

Problem 7. We consider the following nonlinear IVP :

$$\begin{cases} y^{(6)} + y(x)\,y'(x) + y'(x)\,y''(x) = -\sin(x), & 0 \le x \le 1, \\ y(0) = 0,\ y'(0) = 1,\ y''(0) = 0,\ y^{(3)}(0) = -1,\ y^{(4)}(0) = 0,\ y^{(5)}(0) = 1. \end{cases}$$

The exact solution is $y(x) = \sin(x)$. **Table 11** appears comparisons of the numerical solution and absolute errors by presented spline method.

Table 11: The numerical solution and abs errors of problem 7, for *h*=0.1.

x_i	Presented spline methods		
	Exact Solution	Spline Solution	Abs Error
0.1	0.099833416646828152	0.099833416646828151	1.12485975395371E-0018
0.2	0.198669330795061215	0.198669330795061192	2.32019264911898E-0017
0.3	0.295520206661339575	0.295520206661339429	1.46340188231231E-0016
0.4	0.389418342308650492	0.389418342308649923	5.68880879903144E-0016
0.5	0.479425538604203000	0.479425538604201336	1.66389796905919E-0015
0.6	0.564642473395035357	0.564642473395031322	4.03540048599105E-0015
0.7	0.644217687237691054	0.644217687237682483	8.57056396338929E-0015
0.8	0.717356090899522762	0.717356090899506273	1.64889261099199E-0014
0.9	0.783326909627483388	0.783326909627454000	2.93887093480438E-0014
1.0	0.841470984807896507	0.841470984807847220	4.92865839286927E-0014

5. Conclusion

Spline method with three collocation points $x_{k+1/4} = x_k + h/4$, $x_{k+3/4} = x_k + 3h/4$, $x_{k+1} = x_k + h$ is successfully applied in each subinterval $I_k = [x_k, x_{k+1}]$, k=0(1)N-1 to obtain the spline solutions of general linear sixth-order BVPs and nonlinear sixth-order IVPs at every point of the range of integration. The presented spline method is tested on seven problems. Comparisons of the numerical results obtained by the present spline method with others obtained by B-spline collocation methods [1-3], spline collocation method [4], non-polynomial spline methods [5-6], and variation of parameters method [11] reveal that the present method is very effective.

In the next, turbo Pascal program is applied for solving the sixth-order boundary value problem3 in Section 4.

```
PROGRAM Problem3;
  USES WINCRT;   type term=extended;   UVK=array[0..600,0..8] of term;
  VAR y,f:array[0..800] of term;   spl,spl1,spl2,spl3,spl4,spl5,spl6:^UVK;
     U,U1,U2,U3,U4,U5:^UVK; V,V1,V2,V3,V4,v5,v6:^UVK;
     K,K1,K2,K3,K4,k5,k6:^UVK; w,w1,w2,w3,w4,w5,w6:^uvk;
```

```
    JA:^UVK; z:array[1..8] of term;  h,tk,x,mm1,mm,e,b,g,c:term;
     s0,s10,s20,s30,s40,s50,s60,b0,b1,b2,c1,c2,c3:term; i,j,n,m,l,r:INTEGER; FIL:TEXT;
FUNCTION p(a:term;nn:integer):term;
  begin
     if nn=0 then P:=1      else P:=a*P(a,nn-1);
 end;
Function sinh(x:term):term;
   begin     sinh:=(exp(x)-exp(-x))/2;
 end;
Function cosh(x:term):term;
   begin     cosh:=(exp(x)+exp(-x))/2;
 end;
Function exact(x:term):term;
  begin     exact:=1+x*x*x/6+(exp(x)-exp(-x))/2;
 end;
Function exact1(x:term):term;
  begin     exact1:=x*x/2+(exp(x)+exp(-x))/2;
 end;
Function exact2(x:term):term;
   begin     exact2:=x+(exp(x)-exp(-x))/2;
 end;
Function exact3(x:term):term;
   begin  exact3:=1+(exp(x)+exp(-x))/2;
 end;
 Function exact4(x:term):term;
   begin   exact4:=(exp(x)-exp(-x))/2;
 end;
  Function exact5(x:term):term;
    begin   exact5:=(exp(x)+exp(-x))/2;
 end;
 Function exact6(x:term):term;
   begin    exact6:=(exp(x)-exp(-x))/2;
 end;
 Function ss(x:term):term;
 begin
   SS:=s0+x*s10+x*x*s20/2+p(x,3)*s30/6+p(x,4)*s40/24+p(x,5)*s50/120+
     p(x,6)*s60/720+p(x,7)*y[1]/5040+p(x,8)*y[2]/40320+p(x,9)*y[3]/362880;
 end;
 Function ss1(x:term):term;
 begin
SS1:=s10+x*s20+x*x*s30/2+p(x,3)*s40/6+p(x,4)*s50/24+p(x,5)*s60/120+p(x,6)*y[1]/720+
      p(x,7)*y[2]/5040+p(x,8)*y[3]/40320;
 end;
 Function ss2(x:term):term;
 begin
SS2:=s20+s30*x+x*x*s40/2+p(x,3)*s50/6+p(x,4)*s60/24+p(x,5)*y[1]/120+p(x,6)*y[2]/720+
     p(x,7)*y[3]/5040;
 end;
 Function ss3(x:term):term;
  begin
SS3:=s30+x*s40+x*x*s50/2+p(x,3)*s60/6+p(x,4)*y[1]/24+p(x,5)*y[2]/120+p(x,6)*y[3]/720;
 end;
 Function ss4(x:term):term;
  begin    SS4:=s40+x*s50+x*x*s60/2+p(x,3)*y[1]/6+p(x,4)*y[2]/24+p(x,5)*y[3]/120;
 end;
  Function ss5(x:term):term;
  begin    SS5:=s50+x*s60+x*x*y[1]/2+p(x,3)*y[2]/6+p(x,4)*y[3]/24;
 end;
```

```
Function ss6(x:term):term;
  begin    SS6:=s60+x*y[1]+p(x,2)*y[2]/2+p(x,3)*y[3]/6;
 end;
PROCEDURE SOLVE;
 var   k1,lx:integer;  xl:term;   bb:array[0..20,0..20] of term ;
 BEGIN
   for i:=1 to n do
     for j:=n+1 to 2*n do
        begin
         if i=j-n then Ja^[i,j]:=1    else    Ja^[i,j]:=0
        end;   lx:=1 ;
  for k1:=1 to n do
    begin
     if ja^[k1,k1]=0 then
       begin
        for j:=k1+1 to n do
           begin
             if ja^[j,k1]<>0 then
                for i:=1 to 2*n do
                  begin     xl:=ja^[k1,i];   ja^[k1,i]:=ja^[j,i]; ja^[j,i]:=xl;
                   end;      j:=n;
            end;
       end;
     for j:=k1+1 to 2*n do  ja^[k1,j]:=ja^[k1,j]/ja^[k1,k1];
     for j:=lx+1 to n do
       begin
        for i:=k1+1 to 2*n do
           begin     if j<>k1 then
                ja^[j,i]:=ja^[j,i]-ja^[j,k1]*ja^[k1,i];
            end;
        end;   lx:=0;
    end;
     for i:=1 to n do
        for j:=1+n to 2*n do   bb[i,j]:=ja^[i,j];
     for i:=1 to n do  begin y[i]:=0;
          for j:=1+n to 2*n do    y[i]:=y[i]+bb[i,j]*f[j-n];
                     end;
   End; {proc}
 BEGIN {min}
 writeln('y^6-(1+c)y^4+c y^2=cx),[0,1], y(0)=1,y`(0)=1,y`(1)=1/2+cosh(1),y``(1)=1+sinh(1)');
 n:=3;  b0:=7/6+sinh(1);b1:=1/2+cosh(1); b2:=1+sinh(1);
 e:=(5-Sqrt(5))/10;   b:=(5+Sqrt(5))/10;
 Z[1]:=E;Z[2]:=B;Z[3]:=1;  c:=1;  l:=8; h:=(1)/(l);
 s0:=1; s10:=1;  s20:=0; s30:=0; s40:=0;s50:=0; s60:=(1+c)*s40-c*s20+c*0;
 { The Spline Solution of First Initial Value Problem }
 new(Ja);new(U);new(U1);new(U2);new(U3);new(U4);new(U5);
for r:=1 to L do
 begin
 for i:=1 to 3 do begin
 x:=z[i]*h; tk:=(z[i]*h+(r-1)*h);
 f[i]:=-s60+(1+c)*(s40+x*s50+x*x*s60/2)                                    -
c*(s20+x*s30+p(x,2)*s40/2+p(x,3)*s50/6+p(x,4)*s60/24)+c*tk;
 ja^[i,1]:=x -(1+c)*(p(x,3)/6)+c*(p(x,5)/120);
 ja^[i,2]:=p(x,2)/2 -(1+c)*(p(x,4)/24)+c*(p(x,6)/720);
 ja^[i,3]:=p(x,3)/6 -(1+c)*(p(x,5)/120)+c*(p(x,7)/5040);
end;{for i to 3}
SOLVE;
for i:=1 to 3 do
```

```
begin   x:=z[i]*h;
 U^[r,i]:=ss(x);U1^[r,i]:=ss1(x);        U2^[r,i]:=ss2(x);        U3^[r,i]:=ss3(x);        U4^[r,i]:=ss4(x);
U5^[r,i]:=ss5(x);
end;
s0:= U^[r,3];  s10:=U1^[r,3];  s20:=U2^[r,3];  s30:=U3^[r,3];  s40:=U4^[r,3];  s50:=U5^[r,3];
  tk:=r*h;  s60:=(1+c)*s40-c*s20+c*tk;
End;{for l}   { The Spline Solution of Second Initial Value Problem }
  s0:=0; s10:=0; s20:=0;  s30:=1;  s40:=0;  s50:=0; s60:=0;
  new(V); new(V1); new(V2); new(V3); new(V4); new(V5);
 for r:=1 to L do
  begin
  for i:=1 to 3 do begin
  x:=z[i]*h; tk:=(z[i]*h+(r-1)*h);
  f[i]:=-s60+(1+c)*(s40+x*s50+x*x*s60/2)-
c*(s20+x*s30+p(x,2)*s40/2+p(x,3)*s50/6+p(x,4)*s60/24);
  ja^[i,1]:=x -(1+c)*(p(x,3)/6)+c*(p(x,5)/120);
  ja^[i,2]:=p(x,2)/2 -(1+c)*(p(x,4)/24)+c*(p(x,6)/720);
  ja^[i,3]:=p(x,3)/6 -(1+c)*(p(x,5)/120)+c*(p(x,7)/5040);
 end; {for i to 3}
SOLVE;
for i:=1 to 3 do
begin   x:=z[i]*h;
 V^[r,i]:=ss(x);          V1^[r,i]:=ss1(x);     V2^[r,i]:=ss2(x);     V3^[r,i]:=ss3(x);     V4^[r,i]:=ss4(x);
V5^[r,i]:=ss5(x);
end;
s0:=V^[r,3];  s10:=V1^[r,3];  s20:=V2^[r,3];  s30:=V3^[r,3];
s40:=V4^[r,3];  s50:=V5^[r,3];  s60:=(1+c)*s40-c*s20
End;{for l}  { The Spline Solution of Third Initial Value Problem }
  s0:=0; s10:=0; s20:=0; s30:=0; s40:=1; s50:=0; s60:=(1+c)*s40;
  new(K);new(K1);new(K2);new(K3);new(K4);new(K5);
 for r:=1 to L do
  begin
  for i:=1 to 3 do begin
  x:=z[i]*h; tk:=(z[i]*h+(r-1)*h);
  f[i]:=-s60+(1+c)*(s40+x*s50+x*x*s60/2)-
c*(s20+x*s30+p(x,2)*s40/2+p(x,3)*s50/6+p(x,4)*s60/24);
  ja^[i,1]:=x -(1+c)*(p(x,3)/6)+c*(p(x,5)/120);
  ja^[i,2]:=p(x,2)/2 -(1+c)*(p(x,4)/24)+c*(p(x,6)/720);
  ja^[i,3]:=p(x,3)/6 -(1+c)*(p(x,5)/120)+c*(p(x,7)/5040);
 end;{for i to 3}
SOLVE;
for i:=1 to 3 do
begin   x:=z[i]*h;
 K^[r,i]:=ss(x);        K1^[r,i]:=ss1(x);        K2^[r,i]:=ss2(x);    K3^[r,i]:=ss3(x);    K4^[r,i]:=ss4(x);
K5^[r,i]:=ss5(x);
end;
 s0:= K^[r,3];  s10:=K1^[r,3]; s20:=K2^[r,3];s30:=K3^[r,3];
 s40:=K4^[r,3];  s50:=K5^[r,3];  s60:=(1+c)*s40-c*s20;
End;{for l}  { The Spline Solution of Fourth Initial Value Problem }
  s0:=0; s10:=0; s20:=0; s30:=0;  s40:=0; s50:=1; s60:=(1+c)*s40-c*s20 ;
  new(w);new(w1);new(w2);new(w3);new(w4);new(w5);
 for r:=1 to L do
  begin
  for i:=1 to 3 do begin
  x:=z[i]*h; tk:=(z[i]*h+(r-1)*h);
  f[i]:=-s60+(1+c)*(s40+x*s50+x*x*s60/2)-
c*(s20+x*s30+p(x,2)*s40/2+p(x,3)*s50/6+p(x,4)*s60/24);
  ja^[i,1]:=x -(1+c)*(p(x,3)/6)+c*(p(x,5)/120);
```

```
  ja^[i,2]:=p(x,2)/2 -(1+c)*(p(x,4)/24)+c*(p(x,6)/720);
  ja^[i,3]:=p(x,3)/6 -(1+c)*(p(x,5)/120)+c*(p(x,7)/5040);
 end;{for i to 3}  SOLVE;
for i:=1 to 3 do
begin   x:=z[i]*h;
 w^[r,i]:=ss(x);     w1^[r,i]:=ss1(x);    w2^[r,i]:=ss2(x);    w3^[r,i]:=ss3(x);    w4^[r,i]:=ss4(x);
w5^[r,i]:=ss5(x);
end;
s0:= w^[r,3]; s10:=w1^[r,3]; s20:=w2^[r,3];  s30:=w3^[r,3];
s40:=w4^[r,3];  s50:=w5^[r,3];  s60:=(1+c)*s40-c*s20;
End;{for l}  { The Spline Solution of Boundary Value Problem }   n:=3;
  f[1]:=b0-u^[l,3];  f[2]:=b1-u1^[l,3]; f[3]:=b2-u2^[l,3];
  ja^[1,1]:=v^[l,3];   ja^[1,2]:=k^[l,3];   ja^[1,3]:=w^[l,3];
  ja^[2,1]:=v1^[l,3];  ja^[2,2]:=k1^[l,3];  ja^[2,3]:=w1^[l,3];
  ja^[3,1]:=v2^[l,3];  ja^[3,2]:=k2^[l,3];  ja^[3,3]:=w2^[l,3];
  SOLVE;  c1:=y[1]; c2:=y[2];c3:=y[3];
new(spl);new(spl1);new(spl2);new(spl3);new(spl4);new(spl5);new(spl6);
for r:=1  to l do
 for i:=3 to 3 do
  begin
  spl^[r,i]:= U^[r,i]+c1*v^[r,i]+c2*k^[r,i]+c3*W^[r,i];
  spl1^[r,i]:=U1^[r,i]+c1*v1^[r,i]+c2*k1^[r,i]+c3*W1^[r,i];
  spl2^[r,i]:=U2^[r,i]+c1*v2^[r,i]+c2*k2^[r,i]+c3*W2^[r,i];
  spl3^[r,i]:=U3^[r,i]+c1*v3^[r,i]+c2*k3^[r,i]+c3*W3^[r,i];
  spl4^[r,i]:=U4^[r,i]+c1*v4^[r,i]+c2*k4^[r,i]+c3*W4^[r,i];
  spl5^[r,i]:=U5^[r,i]+c1*v5^[r,i]+c2*k5^[r,i]+c3*W5^[r,i];     tk:=r*h;
  spl6^[r,i]:=(1+c)*spl4^[r,3]-c*spl2^[r,3]+tk*c;
  end;
  writeln('   Abs Er S        Abs Er S`      Abs Er S``       Abs Er S```    Xi');
  writeln('==================================================');
 for r:=1  to l do
   begin
    x:=(z[i]*h+(r-1)*h);
    writeln(abs(spl^[r,i]-exact(x)):13,'  ',abs(spl1^[r,i]-exact1(x)):13,
    '  ',abs(spl2^[r,i]-exact2(x)):13,'   ',abs(spl3^[r,i]-exact3(x)):13, '   x',r,'=',x:2:1);
   if r mod 8=0 then  readln;
  end;
  writeln('====================================================');
  writeln('   Abs Er S^4      Abs Er S^5      Abs Er S^6      Xi');
  writeln('====================================================');
 for r:=1  to l do
   begin    x:=r*h;
    writeln(abs(spl4^[r,i]-exact4(x)):13,'    ',abs(spl5^[r,i]-exact5(x)):13,
    '    ',abs(spl6^[r,i]-exact6(x)):13,'     x',r,'=',x:2:1);
   if r mod 10=0 then  readln;
  end;   mm1:=0;
 for r:=1  to l do
 begin   x:=(r*h);
    mm:=abs(spl^[r,i]-exact(x));
    if mm>mm1 then mm1:=mm
 end;writeln('MAX Error S=',mm1:11);
END.
```

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