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Spline Function Method for Solving General Nonlinear Third-Order Boundary Value Problems

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□ ABSTRACT □

In this paper, a numerical method is suggested for solving general a nonlinear third-order boundary value problem (BVP). In this method, the given nonlinear third-order BVP will be transformed into two third-order initial value problems (IVPs), then spline function approximations are applied to both two IVP for finding the Spline solution and its derivatives up to third order of the given BVP. The study shows that the spline solution of the BVP is existent and unique, and the convergence order of the spline method is fourth with a local truncation error $O(h^7)$. The presented algorithm is designed for solving a general BVP, where it is applied to some types of nonlinear third-order differential equations. Comparisons of the results obtained by spline method with other methods show the efficiency and highly accurate of the proposed method.

Key Words: nonlinear boundary value problems, Initial-value problems, Spline function approximations, Collocation points, Convergence analysis.

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طريقة الدالة الشرائحية لحل مسائل القيم الحدية غير الخطية المعممة من المرتبة الثالثة

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ملخّص

في هذا العمل تم اقتراح طريقة عددية لحل مسألة القيم الحدية غير الخطية المعممة من المرتبة الثالثة. هذه الطريقة ستحول مسألة القيمة الحدية المطروحة إلى مسألتين من مسائل القيم الابتدائية، وعندئذ تم تطبيق تقريبات دالة الشريحة على مسألتي القيم الابتدائية للحصول على الحل الشرائحي ومشتقاته حتى المرتبة الثالثة لمسألة القيمة الحدية. تبين الدراسة أنه يوجد حل شرائحي وحيد للمسألة المطروحة، بالإضافة إلى أن مرتبة التقارب للطريقة هي الرابعة وبخطأ مقتطع موضعي $O(h^7)$.

تم وضع خوارزمية للطريقة في الحالة العامة لحل مسألة القيم الحدية، حيث طُبِقتْ لحل بعض مسائل القيم الحدية في المعادلات التفاضلية غير الخطية، وتشير النتائج التي تم الحصول عليها والمقارنات مع بعض الطرائق الأخرى إلى الفعالية الحسابية والدقة العالية للطريقة المقترحة.

الكلمات المفتاحية: مسائل القيم الحدية غير الخطية، مسائل القيم الابتدائية، تقريبات دالة شرائحية، نقاط تجميعية، تحليل التقارب.

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Introduction:

Multi-point boundary value problems arise in a variety of applied mathematics and physics. However, it is usually difficult to obtain closed-form solutions for boundary value problems, especially for nonlinear boundary value problems. In most cases, only approximate solutions can be expected. Some numerical methods have been developed for obtaining approximate solutions to BVPs such as spline methods [**1**-**3**,**5**,**7**] , Padé approximants method [**4**], shooting method [**8**], and finite difference method [**9**]. We present, a numerical method structured on spline function approximations for solving a general third-order boundary value problem of the form

$$U'''(x) = f(x, U, U', U'') \ , \quad a \le x \le b \qquad (1.1)$$

along with the following three cases of boundary conditions

Case I: $U(a) = \alpha,\ U'(a) = \beta,\ U(b) = b_0,$ (1.2a)

Case II: $U(a) = \alpha,\ U'(a) = \beta,\ U'(b) = b_1$, (1.2b)

Case III: $U(a) = \alpha,\ U'(a) = \beta,\ U''(b) = b_2$, (1.2c)

where f is a continuous function on [a, b] and the parameters α, β and b_i, i =0,1,2 are finite real arbitrary constants.

In approximation theory, spline functions occupy an important position having a number of applications, especially in the numerical solutions of boundary-value problems (BVPs). Third-order BVPs are solved by Caglar et al. [**1**] using fourth-degree B-splines, have a first-order accuracy. Khan and Aziz [**2**,**3**] have used a fourth-order method based on quintic splines for solving the third-order linear and nonlinear BVPs of the form $U'''(x) = f(x, U)$, subject to the boundary conditions the case I , also the same problem is solved in [**5**] by using a second-order method based on quartic non-polynomial spline space. Tirmizi et al. [**4**] have solved third-order BVP of the form $U'''(x) = f(x, U')$, this method arises from a four-point recurrence relation involving exponential terms, these being replaced by Padé approximants. An existence and location result for the third-order separated boundary value problem of the form (1.1) with one-sided Nagumo condition have presented by Grossinho et al. in [**6**]. Mahmoud [**7**] has presented quintic spline collocation methods for solving two types of general linear third-order BVPs, those methods are third-order convergent.

1. Importance and Aim of This Work

As we know, it is difficult to give the analytical solutions of problems (1.1)-(1.2c), for that reason, the numerical solutions are very important. The purpose of this work is to provide a numerical treatment for finding the approximate spline solution and its derivatives of the problem at every point of the range of integration. Advantages of the proposed method is extremely simple, quite easy to use, and gives a very good accuracy, and has a computational cost that is cost-effective, because this technique is leaded only to solve a algebraic system of third order, while other numerical methods are leaded to solve algebraic systems of order N, where N the number of grid points.

2. Methodology

Theoretical part: The suggested method in this study is transformed the general nonlinear third-order BVP into third-order initial value problems, after that spline function approximations are directly applied to third-order initial value problems without their reducing into a system of first-order IVPs in ordinary differential equations. Thus, the approximate Spline solution and its derivatives of the nonlinear third-order BVP attain the same order of accuracy for nonlinear third-order IVPs. The proposed spline method when applied to the problem of form (1.1)-(1.2c) is convergent.

Practical part: Numerical results for various problems are compared with those obtained by others. The comparisons show the accuracy, robustness and efficiency of the presented methodology. The computations are accomplished by using *Mathematica Version* 5 and Turbo Pascal under Windows (*TPW* 1.5) in double precision. In consequence, all the problems of form (1.1)-(1.2c) are solvable by using the proposed spline method.

3. Reduction of the BVP into two IVPs

In this section, the nonlinear third-order BVP (1.1) with three cases of the boundary conditions (1.2a)-(1.2c) is reduced into two nonlinear third-order IVPs. This technique is similar to the linear case of shooting method for the third-order BVPs[**7**], except that the solution to a nonlinear problem cannot be simply expressed as a linear combination of the solutions to two IVPs. Instead, it needs to use the solutions to a sequence of initial-value problems of the form:

$$U'''(x) = f(x,U,U',U'') \ , \quad a \le x \le b \qquad (1.3)$$

subject to the initial conditions

$$U(a)=\alpha,\ U'(a)=\beta, U''(a)=t\,, \qquad (1.4)$$

involving a unknown parameter t, to approximate the solution to BVP (1.1) with one of the cases of boundary conditions (1.2a)-(1.2c).

This is performed by choosing the parameter $t = t_k$ in a manner to ensure that

$$\lim_{k\to\infty} U(b,\, t_k) = U(b) = b_0 \qquad (1.5a)$$

is satisfied with boundary conditions, case I : $U(a)=\alpha,\ U'(a)=\beta, U(b)=b_0$,

and that

$$\lim_{k\to\infty} U'(b,\, t_k) = U'(b) = b_1 \qquad (1.5b)$$

is satisfied with boundary conditions, case II : $U(a)=\alpha,\ U'(a)=\beta, U'(b)=b_1$,

and so

$$\lim_{k\to\infty} U''(b,\, t_k) = U''(b) = b_2 \qquad (1.5c)$$

is satisfied with boundary conditions, case III : $U(a)=\alpha,\ U'(a)=\beta, U''(b)=b_2$.

Here, $U(x,t_k)$ denotes the solution of the IVP (1.3)-(1.4) with $t = t_k$, and $U(x)$ denotes the solution of the BVP (1.1)-(1.2). The evaluation starts with a parameter t_0 that determines the initial elevation at which the object is fired from the point (a,α) and along the curve described by the solution to the initial-value problem:

$$U'''(x) = f(x,U,U',U''),\ a \le x \le b\ ,\ U(a)=\alpha,\ U'(a)=\beta, U''(a)=t_0\ . \qquad (1.6)$$

If $U(b,t_0)$ is not sufficiently close to b_0, we can attempt to correct our approximation by choosing another elevation t_1 and so on, until $U(b,t_k)$ is sufficiently close to b_0.

If $U(x,t)$ indicates the solution to the IVP (1.3)-(1.4), the problem associated with determining t so that the following equations are satisfied, respectively:

$$U(b,\,t) - b_0 = 0 \ , \qquad (1.7a)$$

with boundary conditions, case *I,*

$$U'(b,\,t) - b_1 = 0 \ , \qquad (1.7b)$$

with boundary conditions, case *II,*

$$U''(b,\,t) - b_2 = 0 \ , \qquad (1.7c)$$

with boundary conditions, case *III*.

Since (1.7a)-(1.7c) are nonlinear equations, Newton's iteration method will be used to obtain the sequences $\{t_k\}$ of the approximate solutions, when the initial value t_0 is given, the iteration solutions have the following forms, respectively:

$$t_k = t_{k-1} - \frac{U(b,\ t_{k-1}) - b_0}{\frac{d}{dt}U(b,\ t_{k-1})}, \quad k=1,2,\ldots \qquad (1.8a)$$

$$t_k = t_{k-1} - \frac{U'(b,\ t_{k-1}) - b_1}{\frac{d}{dt}U'(b,\ t_{k-1})}, \quad k=1,2,\ldots \qquad (1.8b)$$

$$t_k = t_{k-1} - \frac{U''(b,\ t_{k-1}) - b_2}{\frac{d}{dt}U''(b,\ t_{k-1})}, \quad k=1,2,\ldots \qquad (1.8c)$$

Here, the knowledge of $\frac{d}{dt}U(b,t_{k-1})$, $\frac{d}{dt}U'(b,t_{k-1})$ and $\frac{d}{dt}U''(b,t_{k-1})$ are required. Still this presents a difficulty, since explicit representations for $U(b,t)$, $U'(b,t)$ and $U''(b,t)$ are not known. Suppose that the IVP (1.3)-(1.4) is rewritten, emphasizing that the solution depends on both x and t:

$$\begin{cases} U'''(x,t) = f[x, U(x,t), U'(x,t), U''(x,t)], \quad a \le x \le b \\ U(a,t) = \alpha,\ \ U'(a,t) = \beta,\ \ U''(a,t) = t \end{cases} \qquad (1.9)$$

For this reason, we take the partial derivative of problem (1.9) with respect to t. This implies that:

$$\frac{\partial}{\partial t}U'''(x,t) = \frac{\partial f}{\partial t}[x, U(x,t), U'(x,t), U''(x,t)]$$
$$= \frac{\partial f}{\partial x}[x, U(x,t), U'(x,t), U''(x,t)]\frac{\partial U}{\partial t}(x,t) +$$
$$\frac{\partial f}{\partial U}[x, U(x,t), U'(x,t), U''(x,t)]\frac{\partial U}{\partial t}(x,t) +$$
$$\frac{\partial f}{\partial U'}[x, U(x,t), U'(x,t), U''(x,t)]\frac{\partial U'}{\partial t}(x,t) +$$
$$\frac{\partial f}{\partial U''}[x, U(x,t), U'(x,t), U''(x,t)]\frac{\partial U''}{\partial t}(x,t)$$

Since x and t are independent, then we obtain

$$\begin{aligned} \frac{\partial}{\partial t}U'''(x,t) = {} & \frac{\partial f}{\partial U}[x, U(x,t), U'(x,t), U''(x,t)]\frac{\partial U}{\partial t}(x,t) + \\ & \frac{\partial f}{\partial U'}[x, U(x,t), U'(x,t), U''(x,t)]\frac{\partial U'}{\partial t}(x,t) + \\ & \frac{\partial f}{\partial U''}[x, U(x,t), U'(x,t), U''(x,t)]\frac{\partial U''}{\partial t}(x,t), \quad a \le x \le b \end{aligned} \qquad (1.10)$$

The initial conditions of problem (1.9) are satisfied as follow:

$$\frac{\partial}{\partial t}U(a,t) = 0,\ \ \frac{\partial}{\partial t}U'(a,t) = 0 \text{ and } \frac{\partial}{\partial t}U''(a,t) = 1. \qquad (1.11)$$

By using $W(x,t)$ to denote $\frac{\partial}{\partial t}U(x,t)$ and assume that the order of differentiation of x and t can be reversed, equations (1.10)-(1.11) transform to the initial-value problem:

$$\begin{cases} W''' = \frac{\partial f}{\partial U}(x,U,U',U'')W(x,t) + \frac{\partial f}{\partial U'}(x,U,U',U'')\,W'(x,t) + \frac{\partial f}{\partial U''}(x,U,U',U'')W''(x,t), \\ W(a,t)=0,\ W'(a,t)=0,\ W''(a,t)=1 \quad , \quad a \le x \le b, \end{cases} \quad (1.12)$$

Now, it is required numerical method for solving the both IVPs (1.9) and (1.12), and then the iteration relations (1.8a)-(1.8c) will be known, and can rewrite them as follow:

$$t_k = t_{k-1} - \frac{U(b,t_{k-1}) - b_0}{W(b,t_{k-1})}, \quad k=1,2,\ldots \quad (1.13a)$$

$$t_k = t_{k-1} - \frac{U'(b,t_{k-1}) - b_1}{W'(b,t_{k-1})}, \quad k=1,2,\ldots \quad (1.13b)$$

$$t_k = t_{k-1} - \frac{U''(b,t_{k-1}) - b_2}{W''(b,t_{k-1})}, \quad k=1,2,\ldots \quad (1.13c)$$

In practice, these initial-value problems are not likely to be solved exactly. In the next section, spline function approximations can be applied to obtain the numerical solutions of the both nonlinear third- IVPs (1.9) and (1.12).

Applying Spline Approximations for Non-linear IVPs:

Consider the uniform grid partition $\Delta :\equiv \{a = x_0 < x_1 < \cdots < x_n < x_{n+1} < \cdots < x_N = b\}$ of the interval $[a,\, b]$ with mesh size $h = (b-a)/N$ and grid points $x_n = a + n\,h,\ n = 0,1,\ldots,N$. The Spline function approximation $S(x)$ to the function $U(x,t)$ at the grid points is given by the piecewise expression:

$$S(x) = \begin{cases} S_0(x), & if\ x \in [x_0, x_1], \\ S_n(x), & if\ x \in [x_n, x_{n+1}], \quad n = 1,\ldots,N-2, \\ S_{N-1}(x), & if\ x \in [x_{N-1}, x_N], \end{cases} \quad (2.1)$$

where

$$S_n(x) = \sum_{i=0}^{3} \frac{(x-x_n)^i}{i!} S_n^{(i)} + \sum_{i=4}^{6} \frac{(x-x_n)^i}{i!} C_{n,i-3}, \ x \in [x_n, x_{n+1}],\ n = 0,\ldots,N-1. \quad (2.2)$$

By applying the Spline approximation (2.2) and its derivatives with respect to x, $S_n'(x)$, $S_n''(x)$, $S_n'''(x)$ on three collocation points $x_{n+z_j} = x_n + z_j\,h$, (j=1,2,3), into the third-order IVP (1.9), in each subinterval $\mathrm{I}_n = [x_n,\, x_{n+1}]$, n=0(1)N-1, we have

$$S_n'''(x_{n+z_j}) = f[x_{n+z_j}, S_n(x_{n+z_j}), S_n'(x_{n+z_j}), S_n''(x_{n+z_j})],\ j=1,2,3 \quad (2.3a)$$

with the initial conditions

$$S_0(a) = \alpha,\ S_0'(a) = \beta,\ \ S_0''(a) = t, \quad (2.3b)$$

where $S_0(a) = U(a,t) = \alpha,\ S_0'(a) = U'(a,t) = \beta$ and $S_0''(a) = U''(a,t) = t$, the other coefficient $S_0'''(a) = U'''(a,t)$ is determined by the derivation of equation (1.9), where

$$S_n(x_{n+z_j}) = \sum_{i=0}^{3} \frac{(h z_j)^i}{i!} S_n^{(i)} + \sum_{i=4}^{6} \frac{(h z_j)^i}{i!} C_{n,i-3},\ \ x_{n+z_j} \in [x_n, x_{n+1}],\ \ j = 1,2,3, \quad (2.4)$$
$$n = 0,\ldots,N-1,$$

The first three coefficients $C_{n,1}, C_{n,2}, C_{n,3}$ are computed from the nonlinear system (2.3a) by using the initial conditions (2.3b) for n=0, or from the previous step if n>1.

Similarly, it is required to find the approximate spline solution to the initial-value problem (1.12), to do so, we take the following spline approximation:

$$S_{W,n}(x) = \sum_{i=0}^{3} \frac{(x-x_n)^i}{i!} S_{W,n}^{(i)} + \sum_{i=4}^{6} \frac{(x-x_n)^i}{i!} \tilde{C}_{n,i-3}, \; x \in [x_n, x_{n+1}], \; n = 0,\ldots,N-1, \tag{2.5}$$

by applying the approximation (2.5) into (1.12) to be satisfied at collocation points $x_{n+z_j} = x_n + z_j h$, j=1(1)3, yields:

$$S'''_{W,n}(x_{n+z_j}) = \frac{\partial f}{\partial S_{U,n}} S_{W,n}(x_{n+z_j}) + \frac{\partial f}{\partial S'_{U,n}} S'_{W,n}(x_{n+z_j}) + \frac{\partial f}{\partial S''_{U,n}} S''_{W,n}(x_{n+z_j}) \tag{2.6a}$$

with the initial values:

$$S_{W,0}(a) = S'_{W,0}(a) = 0, \quad S''_{W,0}(a) = 1, \tag{2.6b}$$

where

$$S_{W,n}(x_{n+z_j}) = \sum_{i=0}^{3} \frac{(h z_j)^i}{i!} S_{W,n}^{(i)} + \sum_{i=4}^{6} \frac{(h z_j)^i}{i!} \tilde{C}_{n,i-3}, \; x_{n+z_j} \in [x_n, x_{n+1}], \; j = 1,2,3, \quad n = 0,\ldots,N-1, \tag{2.7}$$

and

$$0 < z_1 < z_2 < z_3 = 1 \tag{2.8}$$

Again, the first three coefficients $\tilde{C}_{n,1}, \tilde{C}_{n,2}, \tilde{C}_{n,3}$ are computed from the nonlinear system (2.6a) by using the initial conditions (2.6b) for n=0, or from the previous step if n>1.

By finding the numerical spline solutions of (2.3a)-(2.3b) and (2.6a)-(2.3b), and substituting $S_{N-1}(x_N, t_{k-1})$, $S_{W,N-1}(x_N, t_{k-1})$ and their derivatives into the iteration relations (1.13a)-(1.13c), we have

$$t_k = t_{k-1} - \frac{S_{N-1}(x_N, t_{k-1}) - b_0}{S_{W,N-1}(x_N, t_{k-1})}, \quad k=1,2,\ldots \tag{2.9a}$$

$$t_k = t_{k-1} - \frac{S'_{N-1}(x_N, t_{k-1}) - b_1}{S'_{W,N-1}(x_N, t_{k-1})}, \quad k=1,2,\ldots \tag{2.9b}$$

$$t_k = t_{k-1} - \frac{S''_{N-1}(x_N, t_{k-1}) - b_2}{S''_{W,N-1}(x_N, t_{k-1})}, \quad k=1,2,\ldots \tag{2.9c}$$

1. A unique Solution

As previous, the numerical solution of the nonlinear third-order BVP (1.1) will transform to the two solutions of initial value problems of the form:

$$U'''(x) = f(x, U(x), U'(x), U''(x)), \; x \in [a,b] \tag{2.10a}$$

$$U(a) = \alpha, \; U'(a) = \beta \text{ and } U''(a) = t \; . \tag{2.10b}$$

Suppose that $f : [a, b] \times C[a, b] \times C^1[a, b] \times C^2[a, b] \to R$ is an enough smooth function satisfying the following Lipschitz condition in respect to the last three arguments:

$$| f(x, y_1, y_2, y_3) - f(x, u_1, u_2, u_3) | \le L(| y_1 - u_1 | + | y_2 - u_2 | + | y_3 - u_3 |), \quad \forall \, (x, y_1, y_2, y_3), (x, u_1, u_2, u_3) \in [a, b] \times R^3 \tag{2.11}$$

where the constant L is called a Lipschitz constant for f.

These conditions assure the existence of a unique solution $U(x)$ of problem (2.10).

By applying the Spline approximations (2.2) and its derivatives into the problem (2.10), using three collocation points $x_{n+z_j} = x_n + z_j h$, (j=1,2,3), we obtain the system

$$S_n''' + (h z_j)C_{n,1} + \frac{(h z_j)^2}{2!} C_{n,2} + \frac{(h z_j)^3}{3!} C_{n,3} =$$
$$f(x_{n+z_j}, S(x_{n+z_j}), S'(x_{n+z_j}), S''(x_{n+z_j})) \ , \quad j = 1,2,3, \qquad (2.12a)$$
$$n = 0,\dots, N-1,$$

$$S_0(a) = \alpha, \quad S_0'(a) = \beta \text{ and } S_0''(a) = t \qquad (2.12b)$$

We rewrite (2.12a) in the matrices formula:

$$A\overline{C}_n = \overline{S}_n + \overline{F}_n \qquad (2.13)$$

where

$$A = \begin{bmatrix} z_1 h & \frac{z_1^2}{2!}h & \frac{z_1^3}{3!}h \\ z_2 h & \frac{z_2^2}{2!}h & \frac{z_2^3}{3!}h \\ h & \frac{1}{2!}h & \frac{1}{3!}h \end{bmatrix}, \ \overline{C}_n = \begin{bmatrix} C_{n,1} \\ hC_{n,2} \\ h^2 C_{n,3} \end{bmatrix}, \ \overline{S}_n = \begin{bmatrix} -S_n^{(3)} \\ -S_n^{(3)} \\ -S_n^{(3)} \end{bmatrix}, \ \overline{F}_n = \begin{bmatrix} f_{n+z_1} \\ f_{n+z_2} \\ f_{n+1} \end{bmatrix},$$

$f_{n+z_j} = f(x_{n+z_j}, S(x_{n+z_j}), S'(x_{n+z_j}), S''(x_{n+z_j}))$, j=1,2,3.

Theorem: Assume that the function $f \in C^2([\mathrm{a,b}] \times \Re \times \Re \times \Re)$ satisfies Lipschitz condition, and if

$$h < \frac{1152}{139441L} \qquad (2.14)$$

then the spline approximation solution $S(x)$ of the problem (2.10) is existent and unique for $z_1 = 1/4,\ z_2 = 3/4, z_3 = 1$, where L is a Lipschitz constant for f.

Proof. From the relation (2.13), it suffices to prove that the vector $\overline{C}_n$ can be uniquely determined for arbitrary given $\overline{S}_n$. Let $\overline{C}_n^1, \overline{C}_n^2 \in R^3$, then from the relation (2.13), we can write

$$\overline{C}_n^1 = A^{-1}\underline{S}_n + A^{-1}\overline{F}_n^1 \text{ and } \overline{C}_n^2 = A^{-1}\underline{S}_n + A^{-1}\overline{F}_n^2$$

Thus $\overline{C}_n^1$ and $\overline{C}_n^2$ can be written in the form

$$\overline{C}_n^1 = \overline{Q}_n(C_{n,1}^1, C_{n,2}^1, C_{n,3}^1, h) \text{ and } \overline{C}_n^2 = \overline{Q}_n(C_{n,1}^2, C_{n,2}^2, C_{n,3}^2, h)$$

Applying $\|.\|_1$, Lipschitz condition and using Mathematica, we get

$$\| \overline{Q}_n(\overline{C}_n^1) - \overline{Q}_n(\overline{C}_n^2) \| = \| (A^{-1}\underline{S}_n + A^{-1}\overline{F}_n^1) - (A^{-1}\underline{S}_n + A^{-1}\overline{F}_n^2) \|$$
$$= \| A^{-1} \| . \| \overline{F}_n^1 - \overline{F}_n^2 \| \le$$

$$\| A^{-1} \| .\{ L_1(\frac{13}{16}h^2 + \frac{23}{96}h^3 + \frac{169}{3072}h^4) | C^1_{n,1} - C^2_{n,1} | +$$
$$L_2(\frac{23}{96}h^2 + \frac{16}{3072}h^3 + \frac{317}{30720}h^4) | C^1_{n,2} - C^2_{n,2} | +$$
$$L_3(\frac{169}{3072}h^2 + \frac{317}{30720}h^3 + \frac{2413}{1474560}h^4) | C^1_{n,3} - C^2_{n,3} |\}$$
$$\leq \| A^{-1} \| .L.(\frac{13}{16}h^2 + \frac{23}{96}h^3 + \frac{169}{3072}h^4).$$
$$\{| C^1_{n,1} - C^2_{n,1} | + | C^1_{n,2} - C^2_{n,2} | + | C^1_{n,3} - C^2_{n,3} |\}$$
$$\leq (\frac{328}{3h})(\frac{3041h^2}{3072} L)\{| C^1_{n,1} - C^2_{n,1} | + | C^1_{n,2} - C^2_{n,2} | + | C^1_{n,3} - C^2_{n,3} |\}$$
$$\leq L\frac{139441}{1152}h.\{ | C^1_{n,1} - C^2_{n,1} | + | C^1_{n,2} - C^2_{n,2} | + | C^1_{n,3} - C^2_{n,3} |\} ,$$

where

$L = \max(L_1, L_2, L_3)$, $H_1 = \max(H_1, H_2, H_3)$, for all $h \in]0,1[$, and

$$H_1 = (\frac{13}{16}h^2 + \frac{23}{96}h^3 + \frac{169}{3072}h^4), \; H_2 = (\frac{23}{96}h^2 + \frac{196}{3072}h^3 + \frac{317}{30720}h^4)$$
$$H_3 = (\frac{196}{3072}h^2 + \frac{317}{30720}h^3 + \frac{2413}{1474560}h^4),$$

$H_1 \leq \frac{3041}{3072}h^2$, for all $h \in]0,1[$, and $\| A^{-1} \| = \frac{1}{h}(\frac{328}{3})$.

Thus, the function $\overline{Q}_n$ defines a contraction mapping if $hL\frac{139441}{1152} < 1$ which satisfies (2.14). Hence there exists a unique $\overline{C}_n$ that satisfies $\overline{C}_n = \overline{Q}_n(C_{n,1}, C_{n,2}, C_{n,3}, h)$ which may be found by iterations, $\overline{C}_n^{p+1} = \overline{Q}_n(\overline{C}_n^p, h)$, p=0,1,2,… .

The proof of the Theorem is now complete.

Convergence Analysis:

We assume that $U(x) \in C^7[a,b]$, the unique solution of the third-order BVP (1.1) and $S_n(x)$ be a spline approximation to $U(x)$, also $T = (\overline{\tau}_n)$ is a 3-dimensional column vector. Here, the vector $\overline{\tau}_n$ is the local truncation error.

Applying the Spline approximation $S_n(x)$ on three collocation points $x_{n+z_j} = x_n + z_j h$, (j=1,2,3), and setting $U(x_{n+z_j}) \cong S_n(x_{n+z_j})$, and $U^{(m)}(x_n) = U_n^{(m)} \cong S_n^{(m)}(x_n)$, ($m$=0,…,3), for $z_1 = 1/4$, $z_2 = 3/4, z_3 = 1$, the local truncation error is formulated by:

$$\overline{\tau}_n = \mathrm{M}\overline{C}_n + \overline{\Psi}_n, \quad n=0,\ldots,N \qquad (3.1)$$

where

$$\overline{\Psi}_n=\begin{bmatrix} U_n+\frac{h}{4}U'_n+\frac{h^2}{32}U''_n+\frac{h^3}{384}U'''_n-U(x_n+\frac{1}{4}h) \\ U_n+\frac{3.h}{4}U'_n+\frac{9h^2}{32}U''_n+\frac{9h^3}{128}U'''_n-U(x_n+\frac{3}{4}h) \\ U_n+hU'_n+\frac{h^2}{2}U''_n+\frac{h^3}{6}U'''_n-U(x_n+h) \end{bmatrix},$$

$$\mathrm{M}=\begin{bmatrix} (\frac{1}{4^4})\frac{h^4}{4!} & (\frac{1}{4^5})\frac{h^4}{5!} & (\frac{1}{4^6})\frac{h^4}{6!} \\ (\frac{3}{4})^4\frac{h^4}{4!} & (\frac{3}{4})^5\frac{h^4}{5!} & (\frac{3}{4})^6\frac{h^4}{6!} \\ \frac{h^4}{4!} & \frac{h^4}{5!} & \frac{h^4}{6!} \end{bmatrix},\quad \overline{C}_k=\begin{bmatrix} C_{n,1} \\ hC_{n,2} \\ h^2C_{n,3} \end{bmatrix}$$

On the other hand, from the system (2.13), we get

$$\overline{C}_n=A^{-1}\hat{U}_n+A^{-1}\hat{F}_n \tag{3.2}$$

where

$$Det(A)=\frac{3h^3}{2048}\neq 0,\quad A^{-1}=\begin{bmatrix} \frac{8}{h} & -\frac{8}{3h} & \frac{1}{h} \\ -\frac{112}{3h} & \frac{80}{3h} & -\frac{32}{3h} \\ \frac{64}{h} & -\frac{64}{h} & \frac{32}{h} \end{bmatrix}.$$

$$\hat{U}_n=\begin{bmatrix} -U_n^{(3)} \\ -U_n^{(3)} \\ -U_n^{(3)} \end{bmatrix},\quad \hat{F}_n=\begin{bmatrix} U'''(x_{n+z_1}) \\ U'''(x_{n+z_2}) \\ U'''(x_{n+1}) \end{bmatrix}.$$

Using Taylor's expansions for the functions $U^{(m)}(x), m=0,\dots,3$ about x_n, in the relation (3.2) and substituting it into (3.1), the local truncation error at the nth step is yielded, as follows:

$$\overline{\tau}_n=\mathrm{M}(A^{-1}\hat{U}_n+A^{-1}\hat{F}_n)+\overline{\Psi}_n=\begin{bmatrix} \frac{103}{165150720}U^{(7)}(x_n).h^7 \\ \frac{81}{18350080}U^{(7)}(x_n).h^7 \\ \frac{1}{322560}U^{(7)}(x_n).h^7 \end{bmatrix}\equiv \mathrm{O}(h^7),\ n=0,1,..N \tag{3.3}$$

where

$$U(x)=\sum_{i=0}^{6}\frac{(x-x_n)^i}{i!}U^{(i)}(x_n)+O(h^7),\ x\in[x_n,x_{n+1}],$$

Note from the relation (3.3) that the proposed Spline method is exact for expansions of the solution of degree ≤ 6, hence, global truncation error is $\|T\|_\infty=N.O(h^7)\equiv O(h^6)$.

Consequently, we have obtained the following: let $U\in C^7[a,b]$ be Lipschitz continuous, then the spline approximation $S(x)$ converges to the solution $U(x)$ of the nonlinear third-order BVP as $h\to 0$ for $z_1=1/4$, $z_2=3/4$, $z_3=1$ and

$$\lim_{h\to 0}S^{(m)}(e)=U^{(m)}(e),\ m=0,\dots,3,\ e=a,b\ . \tag{3.4}$$

Furthermore, the convergence order is furth, i.e., we have

$$|U^{(m)}(x)-S^{(m)}(x)|<C_mh^{7-m},\ m=0,\dots,3\ . \tag{3.5}$$

■ **Algorithm: The Spline Method for solving the nonlinear third-order BVP (1.1)**

We use the following notations elfa= α , beta= β .

INPUTS:

Input (a,b, elfa, beta, t_k, b_0 or b1 else b2): the boundary conditions and constants.

Input (C_1,C_2,C_3), the initial approximation vector $C=(C_1,C_2,C_3)^T$.

Input ($(\tilde{C}_1,\tilde{C}_2,\tilde{C}_3)$), the initial approximation vector $\tilde{C}=(\tilde{C}_1,\tilde{C}_2,\tilde{C}_3)^T$.

Input (N), number of subintervals N.

Input (M), maximum number of iterations M .

Tolerance Tol=0.1E-8; the parameters z1=1/4; z2= 3/4; z3=1;

Step 1 Set h=(b-a)/(N);
ki=1;

Step 2 while (ki <= M) do {0} **Steps** 3-22.

Step 3

Begin {<u>The initial conditions</u>}

S0=elfa;
S1=beta;
S2=tk;
S3=*f*(*a*,s0, s1, s2);
Sw0=0;
Sw1=0;
Sw2=1;
Sw3=S'''w(*a*);

Step 4 for n:=1 to N do {1} **steps** 5-19.

{<u>Spline solution (2.3a) for the nonlinear IVP (1.9)</u>}

We set F_n (C) ≡ $S_n'''(x_{n+z_j})-f[x_{n+z_j},S_n(x_{n+z_j}),S_n'(x_{n+z_j}),S_n''(x_{n+z_j})]$ =0.

Step 5 k=1;

Step 6 while k <= M do {1} **steps** 7-11

Step 7 Calculate F_n (C) and J_n (C),
where $J_n(C)_{i,j}=\partial F_{n,i}(C)/\partial C_j$ for $1\le i,j\le 3$.

Step 8 Solve the 3×3 linear system $J_n(C)Y=-F_n(C)$.

Step 9 Set C=C+Y.

Step 10 if ||Y|| < Tol then {substituting (2.4) and its derivations}
spl[n]=S_n(C);
spl1[n]=$S`_n$(C);
spl2[n]=$S``_n$(C);
spl3[n]=$S```_n$(C);

Step 11 Set k:=1+k;

{ end while 1}

{<u>Spline solution (2.6a) for the nonlinear IVP (1.12)</u>}

We put

$$\bar{F}_n(\tilde{C})\equiv S_{W,n}'''(x_{n+z_j})-\frac{\partial f}{\partial S_{U,n}}S_{W,n}(x_{n+z_j})-\frac{\partial f}{\partial S_{U,n}'}S_{W,n}'(x_{n+z_j})-\frac{\partial f}{\partial S_{U,n}''}S_{W,n}''(x_{n+z_j})=0$$

Step 12 k=1;

steps 13 While (k<=M) do {2} **steps** 14-18.

Step 14 Calculate $\bar{F}_n(\tilde{C})$ and $Jw_n(\tilde{C})$, where $Jw_n(\tilde{C})_{i,j} = \partial Sw_{n,i}(\tilde{C}) / \partial \tilde{C}_j$ for $1 \le i, j \le 3$.

Step 15 Solve the 3×3 linear system $Jw_n(\tilde{C})\bar{Y} = -\bar{F}_n(\tilde{C})$.

Step 16 Set $\tilde{C} = \tilde{C} + \tilde{Y}$.

Step 17 if $\|\tilde{Y}\|$< Tol then {substituting (2.7) and its derivations}

x:=z[i]*h;

Sw[n]=Sw$_n$($\tilde{C}$);

Sw1[n]=Sw`$_n$($\tilde{C}$);

Sw2[n]=Sw``$_n$($\tilde{C}$);

Sw3[n]=Sw``$_n$($\tilde{C}$);

Step 18 Set k:=k+1;

{ end while 2}

Step 19 Set {Renewing the initial conditions}

S0=spl[n]; S1=spl1[n];

S2=spl2[n]; S3=spl3[n];

Sw0=Sw[n]; Sw1=Sw1[n];

Sw2=Sw2[n]; Sw3=Sw3[n];

{ end for 1}

Step 20 if abs(spl[l,3]-b0)<=tol then {else (spl1[l,3]-b1) or (spl2[l,3]-b2) }

Step 21 for n:=1 to N do {2}

Set x=a+z[i]*h+(n-1)*h;

Output(x, Spl[n], Spl1[n], Spl2[n], Spl3[n], tk, Ki);

{ end for 2}

Output ('Procedure is complete. ');

STOP.

Step 22 tk:=tk-(spl[N,3]-b0)/Sw[N,3]; {the iterations (2.9b)-(2.9c) are used for the
ki=ki+1; other boundary conditions ((1.2b)-(1.2c)) }

{ end while 0 &&& Procedure completed unsuccessfully}

END.

Results and Discussion:

In this section, four numerical examples are given to demonstrate the order of convergence and the accuracy of the present spline method. Numerical results of examples are obtained from computer programs designed by *TPW* 1.5 in double precision, and figures are plotted by *Mathematica* 5. The local errors and the rate of convergence are computed for presented spline method.

Here, the notations $E_n^N = |S_n^N - S_{2n}^{2N}|$, $E_n'^N = |S_n'^N - S_{2n}'^{2N}|$, $E_n''^N = |S_n''^N - S_{2n}''^{2N}|$, n=1,…,N, are used where E_n^N indicates the nodal difference error, and S_n^N is the spline approximation of the exact solution $U(x_n)$. Moreover, the experimental nodal rate of convergence is given by $Rate = Log_2(E^N / E^{2N})$.

Problem 4.1 Consider the following nonlinear third-order BVP [1-3,5]

$$U''' = -2e^{-3U} + \frac{4}{(1+x)^3}, \quad 0 \le x \le 1,$$

$U(0)=0$, $U(1)=\text{Ln}(2)$, $U'(0)=1$.

The analytical solution is $U(x) = \text{Ln}(1 + x)$.

This problem will be reduced into the following two initial value problems:

$$\begin{cases} U'''(x,t) = -2e^{-3U(x,t)} + \dfrac{4}{(1+x)^3} \ , 0 \le x \le 1, \\ U(0,t) = 0 \ , U'(0,t) = 1 \ , U''(0,t) = t \ . \end{cases}$$

$$\begin{cases} W'''(x,t) = \dfrac{\partial f}{\partial U} W(x,t) = 6e^{-3U(x,t)} U'(x,t) W(x,t) \ , 0 \le x \le 1, \\ W(0,t) = 0 \ , W'(0,t) = 0 \ , W''(0,t) = 1 \ . \end{cases}$$

Corresponding to the boundary conditions, case *I*, it is required that the iteration relation (2.9a) is satisfied, when $t=t_k$. The best maximum error of the non-polynomial spline method[**5**] is 0.200E-7 for *h*=1/256. **Table1** shows the absolute errors of problem4.1, in addition to comparisons of the spline method with other methods [**1-3**]. In **Table 2,** Observed maximum errors of the Problem 4.1 for step size *h* different are compared with other methods[**5,11**] . **Fig.1** plots the Spline approximation solution with the exact solution of problem 4.1, for *h*=0.1. In **Fig. 2,** the absolute error in the Spline solution of prblem4.1 is depicted for *h*=0.1.

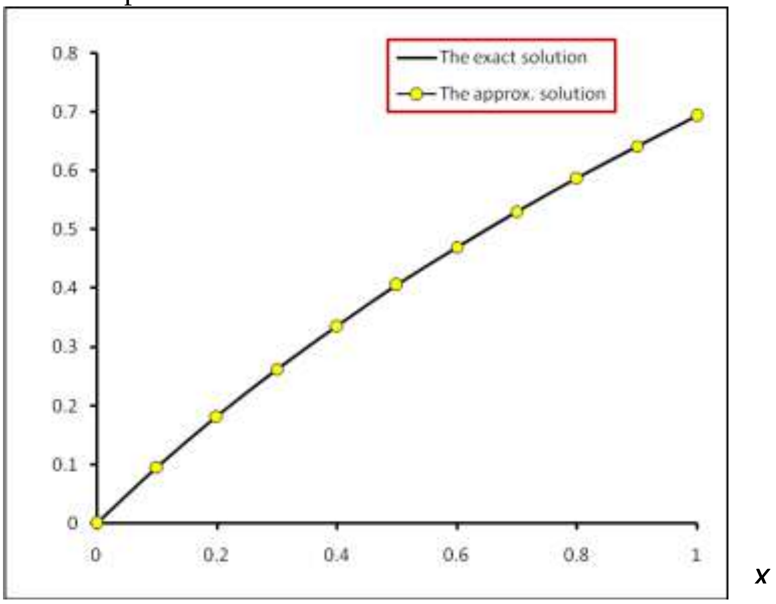


Fig.1. ***the exact solution with the Spline approximation solution of problem 4.1, for h=0.1 .***

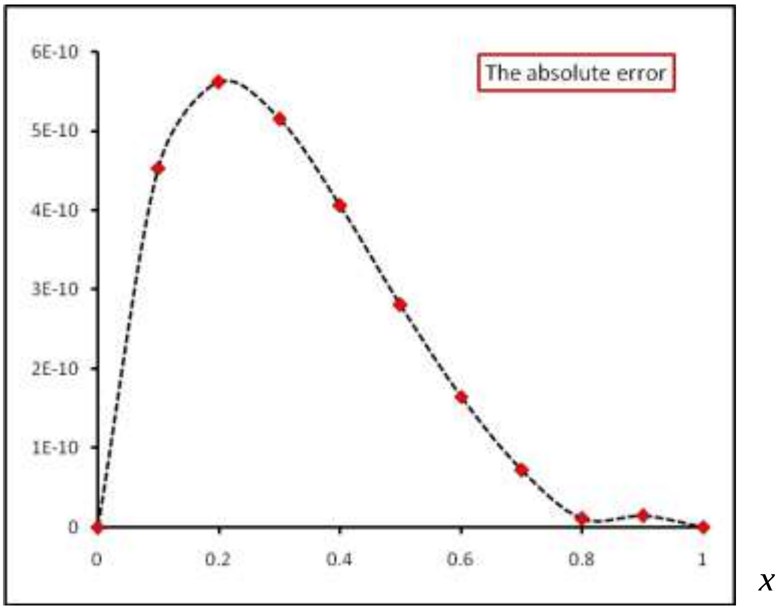


Fig. 2. ***The absolute error in prblem4.1 by the Spline method, for h*=0.1.**

Problem 4.2 Consider the non-linear BVP [**10**]

$$U''' + U(x)U''(x) - [U'(x)]^2 + 1 = 0, \quad 0 \le x \le 1,$$

$U(0)=0, \quad U'(0)=0, \quad U(1)=0.$

Here, it is wanted to solve the two initial value problems:

$$\begin{cases} U'''(x,t) = -U(x,t)U''(x,t) + [U'(x,t)]^2 - 1 \;, \; 0 \le x \le 1, \\ U(0,t) = 0 \;, \; U(0,t) = 0 \;, \; U''(0,t) = t \;. \end{cases}$$

$$\begin{cases} W'''(x,t) = -U''(x,t)W(x,t) + 2U'(x,t)W'(x,t) - U(x,t)W''(x,t) \;, \; 0 \le x \le 1, \\ W(0,t) = 0 \;, \; W'(0,t) = 0 \;, \; W'(0,t) = 1 \;. \end{cases}$$

Also, it is required that the iteration relation (2.9a) is satisfied.

Table 3 shows the spline solution and local errors at the *n*th step of problem 4.2 by present spline method, for *N*=10. By solving the problem 4.2 for *N*=10, 20, 40, the rate of convergence for the spline method is illustrated in **Table 4**. The Spline solution S_i and its derivation S_i' of problem 4.2, for *h*=0.05, are plotted in **Fig.3**. Also in **Fig.4,** the second derivation of Spline solution S_i'' of problem 4.2, for *h*=0.05 .

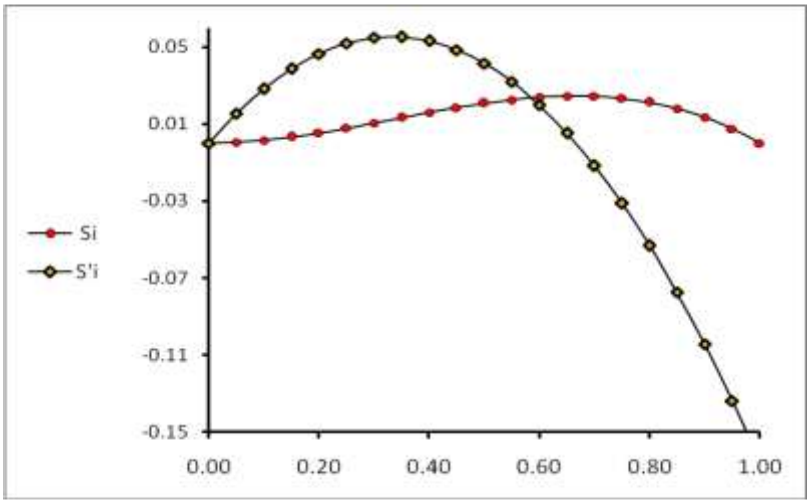


Fig.3. ***The Spline solution S_i and its derivation*** S_i' ***of problem 4.2, for h=0.05 .***

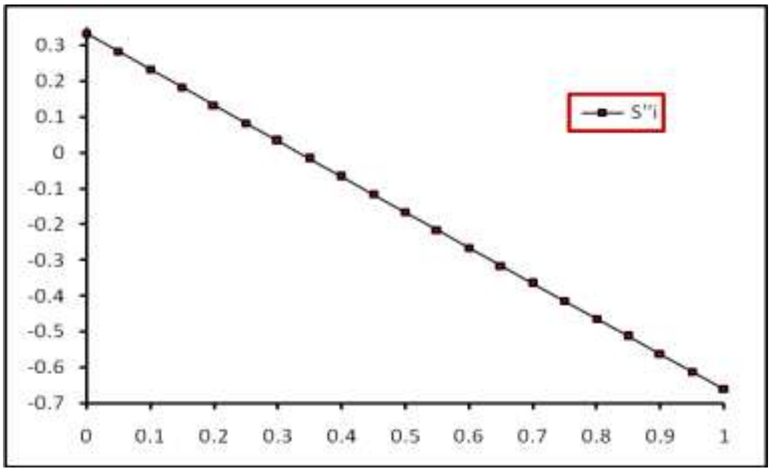


Fig.4. ***The second derivation of Spline solution*** S_i'' ***of problem 4.2, for h=0.05 .***

Problem 4.3 Consider the non-linear BVP [**9**]

$$-\varepsilon U''' - 2U''(x) + 4U'(x) - [U(x)]^2 = f(x), \quad 0 \le x \le 1,$$

$U(0)=1, \quad U'(0)=1, \quad U'(1)=1.$

where

$$f(x) = 1 + 4\left(1 + \frac{1-e^{-2x/\varepsilon}}{2(1-e^{-2/\varepsilon})} - \frac{x}{2}\right) - \left[1 - \frac{\varepsilon(1-e^{-2x/\varepsilon})}{4(1-e^{-2/\varepsilon})} + x\left(1 + \frac{1}{2(1-e^{-2/\varepsilon})}\right) - \frac{x^2}{4}\right]^2 .$$

The solution of this problem is equivalent to solve the two initial value problems:

$$\begin{cases} -\varepsilon U'''(x,t) = 2U''(x,t) - 4U'(x,t) + [U(x,t)]^2 + f(x), \quad 0 \le x \le 1, \\ U(0,t) = 1, \; U'(0,t) = 1, \; U''(0,t) = t . \end{cases}$$

$$\begin{cases} -\varepsilon W'''(x,t) = 2W''(x,t) - 4W'(x,t) + 2U(x,t)W(x,t), \; 0 \le x \le 1, \\ W(0,t) = 0, \; W'(0,t) = 0, \; W''(0,t) = 1 . \end{cases}$$

Moreover, it is required to implement the iteration relation (2.9b), since it fits the boundary conditions, case *II*, it is required that the iteration relation:

$$t_k = t_{k-1} - \frac{S'_{N-1}(x_N, t_{k-1}) - 1}{S'_{W,N-1}(x_N, t_{k-1})}, \quad k=1,2,\ldots$$

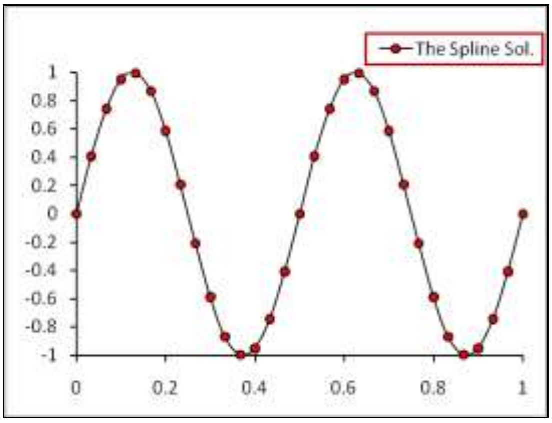
Table 5 displays the spline solution and local errors at the *n*th step of the Spline solution and its derivatives up to third order of the problem 4.3 , for $b=1$, *N*=10, $\varepsilon=1$.

The best maximum error of the computational method [**9**] is 1.636318E-05.

Problem 4.4. Consider the following singular nonlinear problem[**12**]

$$\begin{cases} U'''(x)+\dfrac{2}{x}U'(x)-U''(x)U(x)-16\pi^2U^2(x)=(\dfrac{8\pi}{x}-64\pi^3)\cos(4\pi x) \\ U(0)=0,\quad U'(0)=4\pi,\quad U''(0)=0 \end{cases}$$

Table 6 summarizes the spline solution and its derivatives as well as absolute errors of problem4.4 by presented spline method for *N*=40, *h*=0.0225. In **Fig.5** is plotted the Spline solution S_i and first derivation of S_i' of problem 4.4, for *N*=30.



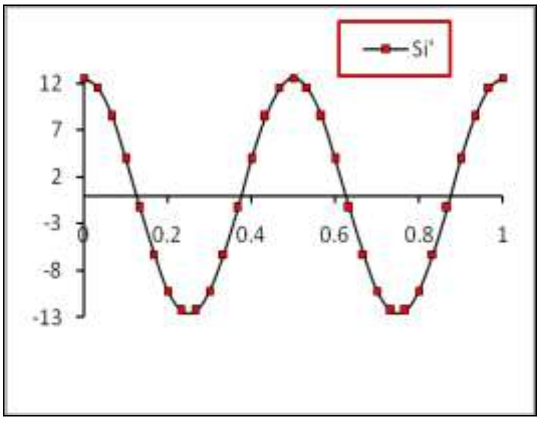


Fig.5: ***The Spline solution S_i and its derivation S_i' of problem 4.4, for N=30, respectively.***

Table 1: ***Comparisons the absolute errors δ_i of Problem 4.1 with other methods***

x_i	B,-spline [1] δ_i	Quintic spline[2] δ_i	Quintic spline[3] δ_i	The Presented Spline Method	
				The Spline Solution S_i	δ_i
0.00	0.00000000	0.0000000	0.0000000	000000000000000	0.000000000000
0.10	0.00985988	5.5798E-6	0.0000056	0.095310180257	4.530420483E-10
0.20	0.0134925	9.44321E-6	0.0000095	0.182321557356	5.616129783E-10
0.30	0.0319635	3.16447E-6	0.0000032	0.262364264983	5.152287485E-10
0.40	0.0456218	1.75366E-5	0.0000175	0.336472237027	4.060893843E-10
0.50	0.0545501	2.90919E-5	0.0000292	0.405465108389	2.810338628E-10
0.60	0.0588473	2.88292E-5	0.0000288	0.470003629411	1.650732883E-10
0.70	0.0585917	1.31489E-5	0.0000132	0.530628251135	7.275957614E-11
0.80	0.0538606	5.0649E-6	0.0000051	0.587786664913	1.091393642E-11
0.90	0.0447198	8.61724E-8	0.0000000	0.641853886158	1.455191522E-11
1.00	6.68056E-6	8.05599E-8	0.0000000	0.693147180559	9.094947017E-13

Table 2: ***Observed maximum errors of the Problem 4.1 for step size h different.***

h	Usmani's method [11]	Quartic non-polynomial spline method [5]	The Present Spline Method
1/8	0.180 E-03	0. 534 E-04	0.2332625 E-08
1/16	0.444 E-04	0. 971 E-05	0.2864908 E-10
1/32	0.120 E-04	0. 199 E-05	0.18189894 E-11
1/64	0.277 E-05	0. 374 E-06	0.45474735 E-12
1/128	0.680 E-06	0. 820 E-07	0.22737367 E-12

Table 3: ***The spline solution and local errors of problem4.2 by presented spline method when, N=10, h=0.1***

x_n	S_n	E_n^N	$E_n'^N$	$E_n''^N$	$E_n'''^N$
0.10	0.00149606947	5.76539E-09	7.8803E-10	5.74810E-09	2.47091E-11
0.20	0.00531781873	1.19943E-08	9.1163E-10	5.0805E-09	2.47082E-11
0.30	0.01046620293	1.80013E-09	3.55389E-10	4.41608E-08	4.94467E-11
0.40	0.01594328099	2.31002E-09	8.34378E-10	3.75784E-08	4.97091E-11
0.50	0.02075232466	2.66541E-09	2.64222E-09	3.1027E-09	3.69507E-11
0.60	0.02389786115	2.80448E-09	5.06813E-09	2.45991E-09	2.47091E-12
0.70	0.02438584791	2.67313E-09	8.09279E09	1.81712E-09	7.41593E-11
0.80	0.02122417681	2.19414E-09	1.17278E-08	1.19906E-09	1.97783E-10
0.90	0.01342370403	1.32497E-09	1.5977E-09	5.68608E-09	4.20275E-10
1.00	0.00000000000	5.56236E-14	2.09525E-09	2.47375E-09	6.92239E-10

Table 4: ***The rate of convergence for presented spline method, with N=10.***

n	$E_n^N = \|S_n^N - S_{2n}^{2N}\|$	$E_n^{2N} = \|S_{2n}^{2N} - S_{4n}^{4N}\|$	Rate of convergence
1	5.76539E-09	1.35207E-10	5.41418
2	1.19943E-08	3.28340E-10	5.19102
3	1.80013E-09	5.40936E-11	5.05650
4	2.31002E-09	7.72546E-11	4.90214
5	2.66541E-09	1.00441E-10	4.72994
6	2.80448E-09	1.15880E-10	4.59703
7	2.67313E-09	1.15881E-10	4.52782
8	2.19414E-09	9.27205E-11	4.56463
9	1.32497E-09	5.40936E-11	4.61436
10	8.56236E-14	2.94445E-15	4.86194

Table 5: ***The spline solution and local errors of problem4.3 by presented spline method for N*=10, *h*=0.1, $\varepsilon = 1$.**

x_n	S_n	E_n^N	$E_n'^N$	$E_n''^N$	$E_n'''^N$
0.10	1.10291561144	7.41820 E-11	2.96646 E-10	7.53051 E-08	1.64208 E-08
0.20	1.21033159287	1.13733 E-10	4.45010 E-10	7.00269 E-08	1.60499 E-08
0.30	1.32052581935	1.28547 E-10	4.94456 E-10	6.34815 E-08	1.50017 E-08
0.40	1.43208833425	1.28547 E-10	8.84561 E-10	5.65422 E-07	1.36024 E-07
0.50	1.54386476268	1.23600 E-10	4.35116 E-09	4.97481 E-07	1.20622 E-07
0.60	1.65490998244	1.13733 E-10	1.60961 E-09	4.34129 E-08	1.05071 E-07
0.70	1.76445019289	1.08785 E-10	9.76911 E-10	3.77082 E-08	9.03612 E-08
0.80	1.87185185968	1.03839 E-10	6.87887 E-10	3.26709 E-08	7.69368 E-08
0.90	1.97659628872	1.08786 E-10	4.88911 E-10	2.83198 E-08	6.47857 E-08
1.00	2.078258809212	1.02324 E-10	3.23602 E-10	2.45867 E-08	5.44887 E-08

Table 6: ***The spline solution, its derivatives and absolute errors of problem4.4 by presented spline method when, N*=40, *h*=0.0225.**

x_n	S_n	S_n'	S_n''	δ_n	δ_n'
0.00	0.00000000000	12.56637061436	0.00000000000	0.00000000000	0.00000000000
0.10	0.95105652158	3.88322222615	-150.184823269	5.28286405E-09	1.48700326E-07
0.20	0.58778528245	-10.16640706019	-92.8193251932	3.01624858E-08	3.24439011E-07
0.30	-0.58778518799	-10.16640709067	92.8193253494	6.43058487E-08	2.93959883E-07
0.40	-0.95105643354	3.88322212521	150.1848222032	8.27582973E-08	4.77541016E-08
0.50	0.00000007415	12.56637043785	-0.00000172981	7.41476726E-08	1.76510399E-07
0.60	0.95105656997	3.88322189917	-150.184824362	5.36771205E-08	1.78282963E-07
0.70	0.58778529418	-10.16640745392	-92.8193255329	4.18858344E-08	6.92935496E-08
0.80	-0.58778521675	-10.16640750109	92.8193253648	3.55447632E-08	1.16461012E-07
0.90	-0.95105650214	3.88322175290	150.1848230498	1.41576724E-08	3.24555123E-07
1.00	-0.00000002590	12.56637019320	-0.00000019857	2.58980856E-08	4.21163376E-07

Conclusion:

The presented algorithm is experimented for solving some problems in nonlinear third-order differential equations. This presented spline method enables us to approximate the solution as well as its derivative at every point of the range of integration, while other methods do not produce. Tables 1-6 and Figures 1-5 are illustrated the applicability, efficiency and very good accuracy of the spline method. Comparisons of our results with the results obtained [**1-3**, **5**, **9**, **11**] show that the presented spline method is better than other methods. Table4 shows the rate of convergence for the spline method when applied to the problem4.2 is at least four, and this is agreed with theoretical study. In consequence, all the problems of form (1.1)-(1.2c) is solvable successfully by using the proposed method.

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