مجلة جامعة تشرين للدراسات والبحوث العلمية _ سلسلة العلوم الأساسية المجلد (29) العدد (1) 2007
Tishreen University Journal for Studies and Scientific Research- Basic Sciences Series Vol. (29) No. (1) 2007

Quintic Spline Collocation Methods for the Solution of Linear Second-Order Boundary Value Problems

Dr. Suliman M. Mahmoud*

(Received 13 / 3 / 2007. Accepted 8/5/2007)

□ ABSTRACT □

This paper presents numerical methods for the solution of linear second-order boundary value problems. These methods are based on C^2-quintic splines, that is, fifth Hermite interpolating polynomials with three collocation points. The error analysis and sufficient conditions of the convergence for the presented methods when applied to BVPs are considered. A study shows that the proposed methods consist of order five for (c_1=1/2, c_2=3/4). Moreover, if:

$$1-3c_1+2c_1^2+c_2(5c_1-c_1^2-3)+c_2^2(2-6c_1+6c_1^2)>0,$$

where $0<c_1<c_2<1$,

then the regions of absolute stability of the methods contain some neighborhood of infinity. They are also A-stable and possess unbounded regions of absolute stability. Four widely applied problems are solved to illustrate the order and stability of the proposed methods. The comparisons of the presented methods with other methods show that our results are more accurate.

Key Words: Linear second-order boundary value problem, Spline collocation methods, Convergence, Consistency, Error analysis, Absolute stability.

* **Assistant Professor, Department of Mathematics, Faculty of Science, Tishreen Universty, Lattakia, Syria. E-mail: Suliman_mmn@yahoo.com.**

مجلة جامعة تشرين للدراسات والبحوث العلمية _ سلسلة العلوم الأساسية المجلد (29) العدد (1) 2007
Tishreen University Journal for Studies and Scientific Research- Basic Sciences Series Vol. (29) No. (1) 2007

طرائق شرائحية مجمعة من المرتبة الخامسة لحل معادلات تفاضلية خطية من المرتبة الثانية بشروط حدية

الدكتور سليمان محمد محمود*

(تاريخ الإيداع 13 / 3 / 2007. قُبِل للنشر في 2007/5/8)

□ الملخّص □

يقدم هذا البحث طرائق عددية لحل مسائل القيم الحدية في المعادلات التفاضلية الخطية من المرتبة الثانية. إن الطرائق المقترحة تعتمد على كثيرات حدود هرمية الشرائحية من الدرجة الخامسة في الفضاء C^2 و تحقق شروط المسألة في ثلاث نقاط مجمعة. حيث يتم تحليل الخطأ لهذه الطرائق بالإضافة إلى وضع الشروط الكافية لتقاربها لدى تطبيقها على مسائل القيم الحدية. تبين الدراسة أن الطرائق المذكورة تكون متجانسة من المرتبة الخامسة لأجل ($c_1=1/2,\ c_2=3/4$)، كما يشير تحليل الاستقرار إلى أنها تكون في حالة A- استقراراً وأن مناطق الاستقرار المطلق تشغل مساحات لانهائية في المستوي العقدي إذا تحققت المتراجعة:

$$1-3c_1+2c_1^2+c_2(5c_1-c_1^2-3)+c_2^2(2-6c_1+6c_1^2)>0 ,$$

علما بأن $0<c_1<c_2<1$.

وقد تم اختبار الطرائق المقترحة باستخدامها لحل أربع مسائل مطبقة على نطاق واسع، وكانت النتائج التي تم التوصل إليها دقيقة بالمقارنة مع طرائق أخرى.

الكلمات المفتاحية : *مسألة القيمة الحدية الخطية من المرتبة الثانية، طرائق شرائحية مجمعة، التقارب، التجانس، تحليل الخطأ، الاستقرار المطلق.*

---

* مدرس في قسم الرياضيات، كلية العلوم، جامعة تشرين، اللاذقية، سورية.

1. Introduction

Linear boundary value problems (BVPs) can be used to model several physical phenomena. For example, a common problem in civil engineering concerns the deflection of a beam of rectangular cross section subject to uniform loading, while the ends of the beam are supported so that they undergo no deflection. This problem is linear second-order two-point BVP.

1.1. Importance of the Work

The main aim of this paper is to study spline collocation methods in order to compute the numerical solution of linear second-order two-point BVP:

$$L[u] \equiv f_2(t)u'' + f_1(t)u' + f_0(t)u = g(t) \quad t \in [a, b], \qquad (1.1)$$

where L is an second order differential operator, $f_0(t)$, $f_1(t)$, $f_2(t)$ and $g(t)$ are given functions and u is the unknown function of t, with on of the three boundary conditions given below:

The boundary conditions of the first kind are:

$$u(a)=\gamma_0,\ u(b)=\gamma_1 \qquad (1.2)$$

The boundary conditions of the second kind are:

$$u'(a)=\gamma_0,\ u'(b)=\gamma_1 \qquad (1.3)$$

The boundary conditions of the third kind, sometimes called *Sturm's boundary conditions*, are:

$$a_0\ u'(a)- a_1\ u(a)=\gamma_0,\ b_0\ u'(b)+ b_1\ u(b)=\gamma_1, \qquad (1.4)$$

where a_0, b_0, a_1 and b_1 are all positive constants.

In (1.1) if $g(t)$=0, the differential equation is called homogeneous; otherwise it is inhomogeneous. Similarly, the boundary conditions are called homogeneous when γ_0, γ_1 are zero; otherwise inhomogeneous.

The contribution is the development and analysis of spline collocation methods with three collocation conditions for the numerical treatment of BVPs.

1.2 A Review of Previous Works

The fist optimal spline collocation methods proposed to solve BVPs were based on cubic splines. For one-dimensional second-order BVPs and uniform partitions, Fyfe [6] proposed a deferred-correction cubic spline method, while [1] and [5] developed and analyzed an extrapolated cubic spline method. Extrapolated and deferred-correction quadratic spline methods, using the midpoints of the uniform partition intervals as collocation points, were proposed and analyzed in [8]. These optimal cubic and quadratic spline collocation methods were extended to two- dimensional second-order elliptic BVPs for rectangular domains in [9] and [4], respectively. Optimal quintic and quartic spline collocation methods [14,10] were developed for one-dimensional fourth-order BVPs on uniform partitions. Christina and Ng [3] developed the optimal quadratic spline collocation methods in [8] to non-uniform partitions. A class of three-point spline collocation methods for solving delay-differential equations is introduced by Mahmoud in [12].

1.3 A Plan of the Paper

The outline of this paper is as follows. In **Section 2**, reducing the linear boundary problems BVPs (1.1)-(1.3) to system of the initial value problems (I.V.Ps.) is presented. Moreover, we introduce the precise description and the formulation of spline collocation methods when applied to BVPs. Sufficient conditions for the convergence of the methods when applied to BVPs are considered in **Section 3**. They show that the proposed methods are consistent with order five for (c_1=1/2, c_2=3/4). The absolute stability analysis is devoted in **Section 4**. Numerical experiments indicate that the regions of absolute stability contain some neighborhood of infinity if $0.803 \le c_1 < c_2 < 1$. In these cases the methods are A-stable and possess unbounded regions of absolute stability. **Section 5** includes several test problems that illustrate the theoretical results. The comparisons of our numerical results with other methods show that our results are more accurate. Finally, conclusions and recommendations are finding in **Section 6**.

2. Description of the Spline Collocation Methods

The aim of this section is to present and analyze quintic spline collocation (QSC) methods for finding a numerical solution of the second order two-point boundary value problem.

2.1 Tow-Point Boundary Value Problem

Consider the two-point BVP (1.1)-(1.3):

$$u'' = p(t)u' + q(t)u + r(t), \quad t \in [a, b], \qquad (2.1a)$$

with either the boundary conditions:

$$u(a)=\gamma_0,\ u(b)=\gamma_1 \qquad (2\ .1b)$$

or the boundary conditions:

$$u'(a)=\gamma_0,\ u'(b)=\gamma_1 \qquad (2.1c)$$

where we assume that $f_2(t)\neq 0$, $p(t)=-f_1(t)/f_2(t)$, $q(t)=-f_0(t)/f_2(t)$ and $r(t)=g(t)/f_2(t)$.

If (2.1a)-(2.1c) satisfies

(i) $p(t)$, $q(t)$, and $r(t)$ are continuous on $[a, b]$,

(ii) $q(t)>0$ on [a, b],

then the problem has a unique solution [2].

First, finding the solution of the linear boundary problem (2.1a)-(2.1b) is assisted by the linear structure of the equation and the use of two special value problems. Suppose that v is the unique solution to the initial value problem (I.V.P.):

$$v'' = p(t)v'(t) + q(t)v(t) + r(t) \quad \text{with } v(a)=\gamma_0 \text{ and } v'(a)=0. \qquad (2.2)$$

In addition, suppose that k is the unique solution to the (I.V.P.):

$$k'' = p(t)k'(t) + q(t)k(t) \quad \text{with } k(a)=0 \text{ and } k'(a)=1. \qquad (2.3)$$

Then the linear combination

$$u(t)=v(t)+c\,k(t), \qquad (2.4)$$

where c is a unknown constant to be determined from the boundary conditions. Note that relation (2.4) is a solution to problem $u'' = p(t)u' + q(t)u + r(t)$, as seen by the computation:

$$\begin{aligned} u''=v''+c\,k'' &= p(t)v'(t) + q(t)v(t) + r(t) + c\,p(t)k'(t) + c\,q(t)k(t) \\ &= p(t)[v'(t) + c\,k'(t)] + q(t)[v(t) + c\,k(t)] + r(t) \\ &= p(t)u'(t) + q(t)u(t) + r(t), \end{aligned}$$

where $u'(t)=v'(t)+c\,k'(t)$.

The solution u in equation (2.4) takes on the boundary values (2.1b):

$$u(a)=v(a)+c\,k(a)=\gamma_0+0=\gamma_0,$$

$$u(b)=v(b)+c\,k(b). \qquad (2.5)$$

Imposing the boundary condition $u(b)=\gamma_1$ in relation (2.5) produces

$c = \dfrac{\gamma_1 - v(b)}{k(b)}$,

and, consequently, when $k(b)\neq 0$, the unique solution to BVP (2.1a)-(2.1b) is:

$u(t)=v(t)+ \dfrac{\gamma_1 - v(b)}{k(b)} k(t)$ (2.6)

In the other hand, the solution of BVP (2.1a) with the boundary conditions of the second kind (2.1c) is obtained in an above similar way; we thus suppose that v is the unique solution to the initial value problem (I.V.P.):

$v'' = p(t)v'(t) + q(t)v(t) + r(t)$ with $v(a)=0$ and $v'(a)=\gamma_0$. (2.7)

Also, suppose that k is the unique solution to the (I.V.P.):

$k'' = p(t)k'(t) + q(t)k(t)$ with $k(a)=1$ and $k'(a)=0$. (2.8)

Then the linear relation:

$u(t)=v(t)+\text{c } k(t)$ (2.9)

is a solution to BVP (2.1a)-(2.1c):

$$u''= p(t)u'(t) + q(t)u(t) + r(t),$$

where $u'(t)=v'(t)+\text{c } k'(t)$,

$u''=v''(t)+\text{c } k''(t)$.

The solution u in equation (2.9) holds the boundary values:

$u'(a)=v'(a)+\text{c } k'(a)=\gamma_0+0=\gamma_0$,

$u'(b)=v'(b)+\text{c } k'(b)$. (2.10)

Imposing the boundary condition $u'(b)=\gamma_1$ in relation (2.10) produces

$c = \dfrac{\gamma_1 - v'(b)}{k'(b)}$.

Hence, if $k'(b)\neq 0$, then the unique solution to BVP (2.1a)-(2.1b) is:

$u(t)=v(t)+ \dfrac{\gamma_1 - v'(b)}{k'(b)} k(t)$. (2.11)

2.2 Formulation of the Spline Approximations

Let $a = t_0 < t_1 < ... < t_N = b$ be a uniform partition of interval $[a,b]$ with $h = (b-a)/\text{N}$ and $t_i = a + i\,h$ for i=0(1)N. Let $S_u \in \text{C}^2$ be the quintic spline collocation approximation of $u(t)$ into each subinterval $\text{I}_i=[t_{i-1}, t_i]$ such that

$$\begin{aligned} S_u(t) = \bar\tau^3&[(6\tau^2 + 3\tau + 1)S^{(0)}_{u,i-1} + (3\tau^2 + \tau)S^{(1)}_{u,i-1} + (\tfrac{1}{2}\tau^2)S^{(2)}_{u,i-1}] \\ &+ \tau^3[(6\bar\tau^2 + 3\bar\tau + 1)S^{(0)}_{u,i} - (3\bar\tau^2 + \bar\tau)S^{(1)}_{u,i} + (\tfrac{1}{2}\bar\tau^2)S^{(2)}_{u,i}] \end{aligned}, \quad (2.12)$$

, (2.13) $S^{(0)}_{u,i-1} = S_u(t_{i-1}),\ S^{(1)}_{u,i-1} = h S'_u(t_{i-1}),\ S^{(2)}_{u,i-1} = h^2 S''_u(t_{i-1})$

where $\tau = (t - t_{i-1})/h \in [0,1],\ \bar\tau = 1 - \tau$.

Differentiating (2.12) two times, we have

$$\begin{aligned} hS'_u(t) = \bar\tau^2&[-30\tau^2 S^{(0)}_{u,i-1} + (1 + 2\tau - 15\tau^2)S^{(1)}_{u,i-1} + (\tau - \tfrac{5}{2}\tau^2)S^{(2)}_{u,i-1}] \\ &- \tau^2[-30\bar\tau^2 S^{(0)}_{u,i} - (1 + 2\bar\tau - 15\bar\tau^2)S^{(1)}_{u,i} + (\bar\tau - \tfrac{5}{2}\bar\tau^2)S^{(2)}_{u,i}] \end{aligned}, \quad (2.14)$$

$$\begin{aligned} h^2 S''_u(t) = \bar\tau&[(120\tau^2 - 60\tau)S^{(0)}_{u,i-1} + (60\tau^2 - 36\tau)S^{(1)}_{u,i-1} + (10\tau^2 - 8\tau + 1)S^{(2)}_{u,i-1}] \\ &+ \tau[(120\bar\tau^2 - 60\bar\tau)S^{(0)}_{u,i} + (36\bar\tau - 60\bar\tau^2)S^{(1)}_{u,i} + (10\bar\tau^2 - 8\bar\tau + 1)S^{(2)}_{u,i}] \end{aligned}. \quad (2.15)$$

2.3 The QSC Methods for BVPs (2.1a)-(2.1c).

First, for solving BVP (2.1a)-(2.1b), it is equivalent to apply the spline approximations (2.12)-(2.15) to two initial value problems (2.2) and (2.3) to obtain the systems:

$$S_v''(t_{i-1+c_j}) = p(t_{i-1+c_j})S_v'(t_{i-1+c_j}) + q(t_{i-1+c_j})S_v(t_{i-1+c_j}) + r(t_{i-1+c_j}), \qquad (2.16)$$

j=1(1)3, i=1(1)N,

with initial values

$S_v(a)=\gamma_0$, $S'_v(a)=0$, $S''_v(a)=q(t_0)\ \gamma_0+r(t_0)$ (2.16a)

and

$$S_k''(t_{i-1+c_j}) = p(t_{i-1+c_j})S_k'(t_{i-1+c_j}) + q(t_{i-1+c_j})S_k(t_{i-1+c_j}), \qquad (2.17)$$

j=1(1)3, i=1(1)N,

with initial values

$S_k(a)=0$, $S'_k(a)=1$, $S''_k(a)=p(t_0)$. (2.17a)

Hence, when $S_k(b)\neq 0$, the spline approximation to BVP (2.1a)-(2.1b) is:

$$S_u(t_{i-1+c_j}) = S_v(t_{i-1+c_j}) + \frac{\gamma_1 - S_v(b)}{S_k(b)}\ S_k(t_{i-1+c_j}) \qquad (2.18)$$

j=1(1)3, i=1(1)N.

However, the approximate spline solution of the BVP (2.1a) with the boundary conditions of the second kind (2.1c) is obtained by applying the spline approximations (2.12)-(2.15) to the two initial value problems (2.7)-(2.8), namely:

$$S_v''(t_{i-1+c_j}) = p(t_{i-1+c_j})S_v'(t_{i-1+c_j}) + q(t_{i-1+c_j})S_v(t_{i-1+c_j}) + r(t_{i-1+c_j}), \qquad (2.19)$$

j=1(1)3, i=1(1)N,

with initial values

$S_v(a)=0$, $S'_v(a)=\gamma_0$, $S''_v(a)=p(t_0)\ \gamma_0+r(t_0)$, (2.19a)

and

$$S_k''(t_{i-1+c_j}) = p(t_{i-1+c_j})S_k'(t_{i-1+c_j}) + q(t_{i-1+c_j})S_k(t_{i-1+c_j}), \qquad (2.20)$$

j=1(1)3, i=1(1)N,

with initial values

$S_k(a)=1$, $S'_k(a)=0$, $S''_k(a)=q(t_0)$. (2.20a)

Therefore, if $S'_k(b)\neq 0$, the spline approximation to BVP (2.1a)-(2.1c) is:

$$S_u(t_{i-1+c_j}) = S_v(t_{i-1+c_j}) + \frac{\gamma_1 - S_v'(b)}{S_k'(b)}\ S_k(t_{i-1+c_j}) \qquad (2.21)$$

j=1(1)3, i=1(1)N,

Note that we used collocation points $t_{i-1+c_j} = t_{i-1} + c_j h$, j=1,2,3, in each subinterval $I_i=[t_{i-1}, t_i]$, i=1(1)N, with $0< c_1< c_2 < c_3=1$, $\bar{c}_j = 1 - c_j$, j=1,2,3.

3. Convergence Analysis and Error Bounds QSC Methods

In this section, we study the order and convergence properties of QSC methods when applied to linear BVP (2.1a)-(2.1b) or BVP (2.1a)-(2.1c). We will assume that $p(t)=q(t)=1$ in (2.1a) and that, without loss of generality.

Applying the spline approximations (2.12)-(2.15) to BVP (2.1a)-(2.1c), we obtain:

$S''_u(t) = S'_u(t) + S_u(t) + r(t)$, (3.1a)

$S_u(t_0) = \gamma_0$, $S_u(t_N) = \gamma_1$, (3.1b)

$S'_u(t_0) = \gamma_0$, $S'_u(t_N) = \gamma_1$. (3.1c)

And letting $S(t)$, $S'(t)$, $S''(t)$ satisfy the collocation conditions:

$$S''_u(t_{i-1+c_j}) = S'_u(t_{i-1+c_j}) + S_u(t_{i-1+c_j}) + r(t_{i-1+c_j}) \ , j=1,2,3,\ i=1(1)N, \quad (3.2)$$

where, $t_{i-1+c_j} \in [t_{i-1}, t_i]$, and $0<c_1<c_2<c_3=1$. One easily has to determine the values $S_{u,i}^{(0)} = S_u(x_i),\ S_{u,i}^{(1)} = hS'_u(x_i),\ S_{u,i}^{(2)} = h^2 S''_u(x_i)$ from the previous step if $i > 1$.

We can write system (3.2) as follows:

$$\begin{aligned}
&[c_j(120\bar{c}_j^2 - 60\bar{c}_j)S_{u,i}^{(0)} + c_j(36\bar{c}_j - 60\bar{c}_j^2)S_{u,i}^{(1)} + c_j(10\bar{c}_j^2 - 8\bar{c}_j + 1)S_{u,i}^{(2)}]/h^2 + \\
&[(-30c_j^2\bar{c}_j^2)S_{u,i}^{(0)} + c_j^2(15\bar{c}_j^2 - 2\bar{c}_j - 1)S_{u,i}^{(1)} + c_j^2(\bar{c}_j - 2.5\bar{c}_j^2)S_{u,i}^{(2)}]/h + \qquad (3.3a)\\
&[c_j^3(-6\bar{c}_j^2 - 3\bar{c}_j - 1)S_{u,i}^{(0)} + c_j^3(3\bar{c}_j^2 + \bar{c}_j)S_{u,i}^{(1)} - (0.5c_j^3\bar{c}_j^2)S_{u,i}^{(2)}] = \\
&\quad [\bar{c}_j(60c_j - 120c_j^2)S_{u,i-1}^{(0)} + \bar{c}_j(36c_j - 60c_j^2)S_{u,i-1}^{(1)} + \bar{c}_j(8c_j - 10c_j^2 - 1)S_{u,i-1}^{(2)}]/h^2 + \\
&\quad [(-30\bar{c}_j^2 c_j^2)S_{u,i-1}^{(0)} + \bar{c}_j^2(1 + 2c_j - 15c_j^2)S_{u,i-1}^{(1)} + \bar{c}_j^2(c_j - 2.5c_j^2)S_{u,i-1}^{(2)}]/h + \\
&\quad [\bar{c}_j^3(6c_j^2 + 3c_j + 1)S_{u,i-1}^{(0)} + \bar{c}_j^3(3c_j^2 + c_j)S_{u,i-1}^{(1)} + (0.5\bar{c}_j^3 c_j^2)S_{u,i-1}^{(2)}] + \\
&\qquad r(t_{i-1+c_j}), \qquad j = 1,2,
\end{aligned}$$

$$S_{u,i}^{(2)} = hS_{u,i}^{(1)} + h^2\ S_{u,i}^{(0)} + h\,r(t_i)\ . \qquad (3.3b)$$

Substituting $S_{u,i}^{(2)} = hS_{u,i}^{(1)} + h^2\ S_{u,i}^{(0)} + h\,r(t_i)$ and $S_{u,i-1}^{(2)} = hS_{u,i-1}^{(1)} + h^2\ S_{u,i-1}^{(0)} + h\,r(t_{i-1})$ into (3.3a)-(3.3b), we get the equivalent recurrence formulae:

$$\begin{cases} A\underline{S}_{u,i} = B\underline{S}_{u,i-1} + h^2\, G\,\underline{R}_{r,i} \\ S_{u,i}^{(2)} = hS_i^{(1)} + h^2 S_i^{(0)} + h^2 r_i(t_i) \end{cases},\ i=1(1)N. \qquad (3.4)$$

where

$$A = \begin{bmatrix}
\begin{aligned}&-c_1[\bar{c}_1^2(12+h^2)(5c_1h + c_1^2h^2 - 20) + \\ &2\bar{c}_1(60 + 8h^2 + 3c_1^2h^2 - c_1h^3) + \\ &2c_1^2h^2 - 2h^2]/2\end{aligned} & \begin{aligned}&-c_1[\bar{c}_1^2(h-6)(5c_1h + c_1^2h^2 - 20) - \\ &2\bar{c}_1(36 - 8h - 2c_1h + c_1h^2 + c_1^2h^2) - \\ &2h + 2c_1h]/2\end{aligned} \\
\begin{aligned}&-c_2[\bar{c}_2^2(12+h^2)(5c_2h + c_2^2h^2 - 20) + \\ &2\bar{c}_2(60 + 8h^2 + 3c_2^2h^2 - c_2h^3) + \\ &2c_2^2h^2 - 2h^2]/2\end{aligned} & \begin{aligned}&-c_2[\bar{c}_2^2(h-6)(5c_2h + c_2^2h^2 - 20) - \\ &2\bar{c}_2(36 - 8h - 2c_2h + c_2h^2 + c_2^2h^2) - \\ &2h + 2c_2h]/2\end{aligned}
\end{bmatrix}$$

$$B=\left[\begin{array}{c:c}
\begin{array}{l}\frac{1}{2}\bar{c}_1c_1^2(12+h^2)(\bar{c}_1^2h^2-5\bar{c}_1h-20)+\\ 2\bar{c}_1c_1(60+8h^2+3\bar{c}_1^2h^2+\bar{c}_1h^2)+\\ \bar{c}_1(\bar{c}_1^2h^2+h-1)h^2\end{array} & \begin{array}{l}\frac{1}{2}\bar{c}_1[c_1^2(h+6)(\bar{c}_1^2h^2-5\bar{c}_1h-20)+\\ 2c_1(36+8h+2\bar{c}_1h+\bar{c}_1h^2+\bar{c}_1^2h^2)+\\ 2\bar{c}_1h-2h]\end{array} \\ \hdashline
\begin{array}{l}\frac{1}{2}\bar{c}_2c_2^2(12+h^2)(\bar{c}_2^2h^2-5\bar{c}_2h-20)+\\ 2\bar{c}_2c_2(60+8h^2+3\bar{c}_2^2h^2+\bar{c}_2h^2)+\\ \bar{c}_2(\bar{c}_2^2h^2+h-1)h^2\end{array} & \begin{array}{l}\frac{1}{2}\bar{c}_2[c_2^2(h+6)(\bar{c}_2^2h^2-5\bar{c}_2h-20)+\\ 2c_2(36+8h+2\bar{c}_2h+\bar{c}_2h^2+\bar{c}_2^2h^2)+\\ 2\bar{c}_2h-2h]\end{array}
\end{array}\right]$$

$$G=\left[\begin{array}{c:c:c:c}
\begin{array}{l}\bar{c}_1(8c_1-10c_1^2-1)+\bar{c}_1^3c_1^2h^2/2+\\ \bar{c}_1^2(c_1-2.5c_1^2)h\end{array} & 1 & 0 & \begin{array}{l}c_1(8\bar{c}_1-10c_1^2-1)+c_1^3\bar{c}_1^2h^2/2+\\ c_1^2(\bar{c}_1-2.5\bar{c}_1^2)h\end{array} \\ \hdashline
\begin{array}{l}\bar{c}_2(8c_2-10c_2^2-1)+\bar{c}_2^3c_2^2h^2/2+\\ \bar{c}_2^2(c_2-2.5c_2^2)h\end{array} & 0 & 1 & \begin{array}{l}c_2(8\bar{c}_2-10c_2^2-1)+c_2^3\bar{c}_2^2h^2/2+\\ c_2^2(\bar{c}_2-2.5\bar{c}_2^2)h\end{array}
\end{array}\right],$$

$\underline{S}_{u,i}=(S_i^{(0)},S_i^{(1)})^T$, $\underline{R}_i=(r_{i-1},r_{i-1+c_1},r_{i-1+c_2},r_i)^T$.

3.1 Matrix Analysis of QSC Methods

Now, we present the properties of the spline collocation matrices which arise from applying the spline method (c_1=1/2, c_2=3/4) of linear system (3.5) with boundary conditions (3.1b) or (3.1c). We can show that the matrices A and B in the linear system (3.5) are non-singular because for the method (c_1=1/2, c_2=3/4), we have

$$A=\begin{bmatrix}\dfrac{(-120h-48h^2-2h^3-h^4)}{64} & \dfrac{(96+12h+8h^2-h^3)}{64}\\ \dfrac{9(12h^3-3h^4-268h^2-240h-1280)}{2048} & \dfrac{9(896-136h+54h^2-3h^3)}{2048}\end{bmatrix},$$

whence

$$|A|=\frac{9(30720-17280h+3040h^2+240h^3-202h^4-39h^5-3h^6)}{32768}\neq 0,\ \forall\ h\in[0,\ 2.38].$$

That is, $|A|\to 135/16$, as $h\to 0$.

Moreover, for the same method (c_1=1/2, c_2=3/4), we get

$$B=\begin{bmatrix}\dfrac{(-120h+48h^2-2h^3+h^4)}{64} & \dfrac{(96-12h+8h^2+h^3)}{64}\\ \dfrac{9(3h^4-28h^3-36h^2-720h-3840)}{2048} & \dfrac{3(-1152-360h-2h^2+3h^3)}{2048}\end{bmatrix},$$

and hence

$$|B|=\frac{3(92160+40320h+3360h^2-720h^3-126h^4+21h^5-h^6)}{32768}\neq 0,\ \forall\ h\in[0,\ 9.68].$$

It is easy to find that $|B|\to 135/16$ as $h\to 0$. Thus, linear system (3.4) with boundary conditions (3.1b) or (3.1c) exists and has a unique spline approximation solution given by:

$$\begin{cases} \underline{S}_{u,i} = A^{-1}B\underline{S}_{u,i-1} + h^2 A^{-1}G\underline{R}_{r,i} \\ S_i^{(2)} = hS_i^{(1)} + h^2 S_i^{(0)} + h^2 r_i(t_i) \end{cases} . \qquad (3.5)$$

3.2 Convergence and Error Bounds of QSC Methods

Here, we need the following definition:

Definition 1:[7] A method is said to consist of order p if $\max_{0\le i\le N} \| d_i \| = O(h^p)$, where d_i is a global discretization error at x_i.

Theorem 1 Let $u \in C^7[a,b]$, then the QSC methods (2.16)-(2.17) or (2.19)-(2.20) are consistent of order five for (c_1=1/2, c_2=3/4).

Proof. We obtain from the system (3.5) the local discretization error:

$$d_i = \begin{bmatrix} u(t_i) \\ hu'(t_i) \end{bmatrix} - A^{-1}B\begin{bmatrix} u(t_{i-1}) \\ hu'(t_{i-1}) \end{bmatrix} - h^2 A^{-1}G[r(t_{i-1}),\ r(t_{i-1+c_1}),\ r(t_{i-1+c_2}),\ r(t_i)]^T, \quad (3.6)$$

where $r(t) = u''(t) - u'(t) - u(t)$.

Using Taylor's expansions for the functions $u(t)$, $u'(t)$ and $u''(t)$ about t_{i-1}, t_{i-1+c_j}, j=1,2,3, and substituting in (3.6), we get:

$$d_i = \begin{bmatrix} \dfrac{h^6(-46080 + 3840h - 4640h^2 + 292h^3 - 60h^4 + 3h^5)}{5760(-30720 + 17250h - 3040h^2 - 240h^3 + 202h^4 + 39h^5 + 3h^6)} \\ \dfrac{h^6(61440 + 15360h + 19840h^2 + 760h^3 + 616h^4 - 6h^5 + 3h^6)}{5760(-30720 + 17250h - 3040h^2 - 240h^3 + 202h^4 + 39h^5 + 3h^6)} \end{bmatrix} u^{(6)}(t_{i-1}) + O(h^7),$$

where

$$u(t) = \sum_{k=0}^{5} \frac{h^k}{k!} u^{(k)}(t_{i-1}) + \frac{h^6}{6!} u^{(6)}(t_{i-1})\tau^6 + O(h^7),\ u \in [t_{i-1}, t_i]. \qquad (3.7)$$

Therefore, from the Taylor's expansions of rational functions about h are:

$$\frac{h^6(-46080 + 3840h - 4640h^2 + 292h^3 - 60h^4 + 3h^5)}{5760(-30720 + 17250h - 3040h^2 - 240h^3 + 202h^4 + 39h^5 + 3h^6)} = \frac{h^6}{3840} + O(h^7),$$

$$\frac{h^6(61440 + 15360h + 19840h^2 + 760h^3 + 616h^4 - 6h^5 + 3h^6)}{5760(-30720 + 17250h - 3040h^2 - 240h^3 + 202h^4 + 39h^5 + 3h^6)} = \frac{h^6}{2880} + O(h^7),$$

whence

$$d_i = \begin{bmatrix} h^6/3840 \\ h^6/2850 \end{bmatrix} u^{(6)}(t_{i-1}) + O(h^7).$$

Thus, $\| d_i \| = \dfrac{h^6}{2304} | u^{(6)}(t_{i-1}) | + O(h^7) = O(h^6)$. Since the proposed methods are exact for polynomials of degree$\le$5, and noting that local discretization error is of order six, we deduce according to Definition 1, that the methods are thus consistent of order at least five. This completes the proof.

Corollary 1. Let $u \in C^7[a,b]$ be Lipschitz continuous, then the spline approximation $S_u(t)$ given by (2.12)-(2.13) converges to the solution $u(x)$ of (2.1) as $h \to 0$ for all c_1, $c_2 \in (0,1)$, with $c_1 \ne c_2$ and

$$\lim_{h\to 0} h^{-j} S_u^{(j)}(t_e) = u^{(j)}(t_e),\ \ j = 0(1)2,\ e=a,b.$$

Furthermore, the convergence order is ≥ five, i.e., we have

$$|u^{(k)}(t_i) - h^{-k} S_u^{(k)}(t_i)| \le C_k h^5,\ k = 0,1,2,\ i = 1(1)\mathrm{N}, \quad (3.7a)$$

and

$$|u^{(3)}(t_i) - h^{-2} S_u^{(3)}(t_i)| \le C_3 h^3,\quad i = 1(1)\mathrm{N}, \quad (3.7b)$$

whenever the initial and boundary conditions satisfy (3.7) (with i=0).

In addition, for $t \ne t_i$, i=1(1)N, estimating the interpolation errors of the spline approximations is as followed:

$$\left|S_{u,i}(t) - u(t)\right| = \frac{\tau^3\,\bar\tau^3}{720} u^{(6)}(t_{i-1})h^6 + O(h^7),\ t \in [t_{i-1}, t_i],$$

$$\left|S'_{u,i}(t) - u'(t)\right| = \frac{\tau^2\,\bar\tau^2(1-2\tau)}{240} u^{(6)}(t_{i-1})h^5 + O(h^6),\ t \in [t_{i-1}, t_i],$$

$$\left|S''_{u,i}(t) - u''(t)\right| = \frac{\tau\,\bar\tau(1-5\tau\,\bar\tau)}{120} u^{(6)}(t_{i-1})h^4 + O(h^5),\ t \in [t_{i-1}, t_i],\ i=1(1)\mathrm{N},$$

where $\tau = (t - t_{i-1})/h \in [0,1]$, $\bar\tau = 1 - \tau$ and,

$$S_u(t) = \bar\tau^3[(6\tau^2 + 3\tau + 1)u(t_{i-1}) + (3\tau^2 + \tau)hu'(t_{i-1}) + (\tfrac{1}{2}\tau^2)h^2u''(t_{i-1})]$$
$$+ \tau^3[(6\tau'^2 + 3\bar\tau + 1)u(t_i) - (3\bar\tau^2 + \bar\tau)hu'(t_i) + (\tfrac{1}{2}\bar\tau^2)h^2u''(t_i)] .$$

Moreover, the following global error estimate holds true:

$$\left|u^{(k)}(t) - S_u^{(k)}(t)\right| \le C_k h^{5-k},\ k = 0(1)4,\ t \in [a,b].$$

4. Absolute Stability of the QSC Methods

Here, we restrict our attention to the linear stability properties of the QSC methods. Applying the proposed methods to the test equation:

$$U'' - \lambda^2 U = 0, \quad (4.1)$$

where $\lambda \in C$ is arbitrary, we get from (2.12)-(2-13) and (2.15) the equations

$$S_{u,i}^{(2)} = zS_{u,i}^{(0)},\ S_{u,i-1}^{(2)} = zS_{u,i-1}^{(0)},\ (z \equiv (h\lambda)^2)$$

and

$$\begin{aligned}
&[c_j(120\bar c_j^2 - 60c'_j) - 3Zc_j\bar c_j(3 - \bar c_j - 3\bar c_j^2 + 2\bar c_j^3) - \tfrac{1}{2}Z^2(c_j^3\bar c_j^3)]S_{u,i}^{(0)}\\
&\quad + [c_j(36\bar c_j - 60\bar c_j^2) + Zc_j^3(\bar c_j + 3\bar c_j^2)]S_{u,i}^{(1)} =\\
&[\bar c_j(60c_j - 120c_j^2) + 3Zc_j\bar c_j(3 + 3c_j - 3c_j^2 + 2c_j^3) + \tfrac{1}{2}Z^2(c_j^3\bar c_j^3)]S_{u,i-1}^{(0)}\\
&\quad + [\bar c_j(36c_j - 60c_j^2) + Z\bar c_j^3(c_j + 3c_j^2)]S_{u,i-1}^{(1)},\qquad j = 1,2.
\end{aligned} \quad (4.2)$$

And after simplifying these expressions, we get

$$C(z).\underline{S}_i = D(z)\underline{S}_{i-1}, \quad (4.3)$$

where

$$C(z) = \begin{bmatrix} 120\bar c_1 - 60 - 3Z(3 - 3\bar c_1 - 3\bar c_1^2 + 2\bar c_1^3) - \tfrac{1}{2}Z^2c_1^2\bar c_1 & 36 - 60\bar c_1 + Zc_1^2(1 + 3\bar c_1) \\ 120\bar c_2 - 60 - 3Z(3 - 3\bar c_2 - 3\bar c_2^2 + 2\bar c_2^3) - \tfrac{1}{2}Z^2c_2^2\bar c_2 & 36 - 60\bar c_2 + Zc_2^2(1 + 3\bar c_2) \end{bmatrix}$$

and

$$D(z)=\begin{bmatrix} 60-120c_1+3Z(3-3c_1-3c_1^2+2c_1^3)+\frac{1}{2}Z^2\bar{c}_1^2c_1 & 36-60c_1+Z\,\bar{c}_1^2(1+3c_1) \\ 60-120c_2+3Z(3-3c_2-3c_2^2+2c_2^3)+\frac{1}{2}Z^2\bar{c}_2^2c_2 & 36-60c_2+Z\,\bar{c}_2^2(1+3c_2) \end{bmatrix}$$

Thus, by definition, $z=(h\lambda)^2$ belongs to the region of absolute stability $\Omega\equiv\Omega_{c_1,c_2}$ of the methods if the eigenvalues $\mu_1=\mu_1(z)$ *and* $\mu_2=\mu_2(z)$ of the generalized eigenvalue problem

$$\mu\, C(z).\underline{x}=D(z).\underline{x}, \quad \underline{x}\neq 0, \tag{4.4}$$

lie inside to the unit disc in the complex plane, i.e. if

$$|\mu_1|,|\mu_2|<1. \tag{4.5}$$

The coefficients of the quadratic (in μ) characteristic equation $\det(\mu\, C(z)-D(z))=0$, are polynomials of degree 3 in Z which makes it almost impossible to find explicit descriptions of $\Omega\equiv\Omega_{c_1,c_2}$ by (4.4)-(4.5) directly. Instead, we first consider the asymptotic behavior as $z\to\infty$. Multiplying the characterizing equation by Z^{-3}, we get

$$\lim_{z\to\infty} Z^{-3}\det(\mu C(z)-D(z))=\begin{vmatrix} -\frac{1}{2}(c_1^2\bar{c}_1\mu+c_1\bar{c}_1^2) & c_1^2(1+3\bar{c}_1)\mu-\bar{c}_1^2(1+3c_1) \\ -\frac{1}{2}(c_2^2\bar{c}_2\mu+c_2\bar{c}_2^2) & c_2^2(1+3\bar{c}_2)\mu-\bar{c}_2^2(1+3c_2) \end{vmatrix}$$

$$=(c_1^2c_2^2)\,\mu^2+(c_1\bar{c}_1+c_2\bar{c}_2+4c_1c_2\bar{c}_2-4c_2\bar{c}_2c_1^2)\,\mu+\bar{c}_1^2\bar{c}_2^2.$$

Thus, some neighborhood of $z\to\infty$ is contained in $\Omega\equiv\Omega_{c_1,c_2}$ if the two solutions $\mu_1=\mu_1(\infty)$, $\mu_2=\mu_2(\infty)$ of the equation:

$$(c_1^2c_2^2)\,\mu^2+(c_1\bar{c}_1+c_2\bar{c}_2+4c_1c_2\bar{c}_2-4c_2\bar{c}_2c_1^2)\,\mu+\bar{c}_1^2\,\bar{c}_2^2=0,$$

hold the inequalities (4.5).

We can find by using Mathematica program that

$$\max(|\mu_1|,|\mu_2|)=1+[(c_1+c_2)\bar{c}_1\bar{c}_2+(\bar{c}_1+\bar{c}_2)c_1c_2+4c_1c_2\,\bar{c}_1\bar{c}_2 \\ -(c_1c_2)^2-(\bar{c}_1\bar{c}_2)^2]/c_1c_2\bar{c}_1\bar{c}_2 \,.$$

Consequently, putting $\bar{c}_j=1-c_j$, j=1,2, for $\max(|\mu_1|,|\mu_2|)<1$, we get the inequality

$$1-3c_1+2c_1^2+c_2(5c_1-c_1^2-3)+c_2^2(2-6c_1+6c_1^2)>0\,, \tag{4.6}$$

which satisfies (4.5).

For various c_1 and c_2 the stability region $\Omega\equiv\Omega_{c_1,c_2}$ was obtained numerically by determining the boundary curve $z=z(\varphi)$ according to $\det(e^{\varphi i}C(z)-D(z))=0$

(cf.[7],Ch.2). Numerical experiments indicate that for $0.803 \le c_1 < c_2 < 1$, the methods will be A-stable independent of the particular choice of the two interior collocation points (cf. Table 1).

Fig.1 depicts the regions of absolute stability for $c_1 = 0.5$ and different c_2. Also, some regions of stability in the case $c_1 = 0.75$ and different c_2 are listed in Fig 2.

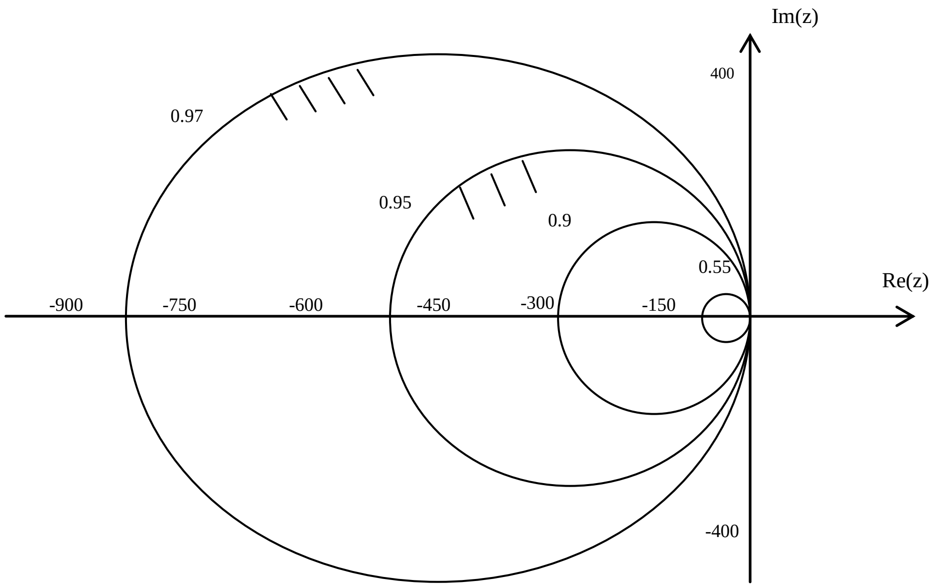

Fig.1: Some regions of absolute stability for $c_1 = 0.5$ and different c_2

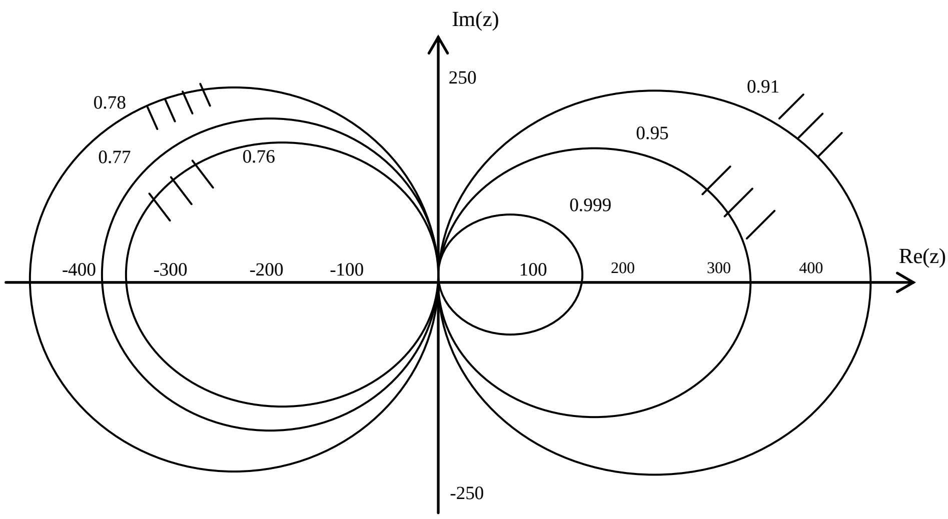

Fig.2: Some regions of absolute stability for $c_1 = 0.75$ and different c_2

Table 1: Some intervals, which determine some unbounded regions of A-stability

$0.5<c_1$	$0.9999\le c_2<1$	$\|\mu_1\|<1$; $\|\mu_2\|<9.9E-15$
$0.60\le c_1$	$0.96<c_2<1$	$\|\mu_1\|<0.9611$; $\|\mu_2\|<1.709E-3$
$0.65\le c_1$	$0.93<c_2<1$	$\|\mu_1\|<0.9494$; $\|\mu_2\|<8.127E-3$
$0.70\le c_1$	$0.90<c_2<1$	$\|\mu_1\|<0.9439$; $\|\mu_2\|<2.402E-3$
$0.75\le c_1$	$0.86<c_2<1$	$\|\mu_1\|<0.9541$; $\|\mu_2\|<3.086E-3$
$0.80\le c_1$	$0.81<c_2<1$	$\|\mu_1\|<0.9786$; $\|\mu_2\|<3.514E-3$
$0.8028\le c_1$	$0.805<c_2<1$	$\|\mu_1\|<0.9990$; $\|\mu_2\|<3.635E-3$
$0.803\le c_1$	$0.803<c_2<1$	$\|\mu_1\|<0.9980$; $\|\mu_2\|<3.629E-3$

5. Numerical Results

In this section, we present numerical results to demonstrate the convergence of the spline collocation methods for BVPs with uniform grids. All computations were carried out in double precision. We have programmed the QSC methods in Mathematica. The experiments below are designed to test the efficiency of the spline methods for linear BVPs. These problems have exact solutions. Thus, we compute their actual errors.

Problem 1. Consider the linear BVP (cf. [3]):

$$u''+u'-u=g,\quad t\in(0,1),$$
$$u(0)=0,\ u(1)=1.$$

The function g is chosen so that $u(t)=t^q$, q>0, is the solution to the problem. In Table2, we compared the absolute error norm of QSC methods with other method.

Problem 2. Solution of the second problem has a boundary layer at the left endpoint. The parameter η controls the sharpness of boundary layer.

$$\{(1+\eta t)u'\}'=0,\quad t\in(0,1),$$
$$u(0)=0,\ u(1)=1.$$

The analytical solution to this problem is $u(t)=\dfrac{\log(1+\eta t)}{\log(1+\eta)}$. Fig. 3 plots u for various values of η. The function u with large η increases very sharply near t=0. In order to accurately capture this property in the approximate solution, appropriate collocation points of the QSC methods should be used. Fig. 4 illustrates both the approximate solution and the exact solution for N=32, with η=100. Table 3 shows the comparisons of absolute error norms of QSC methods with other method for various values of η =1, 100, 10000.

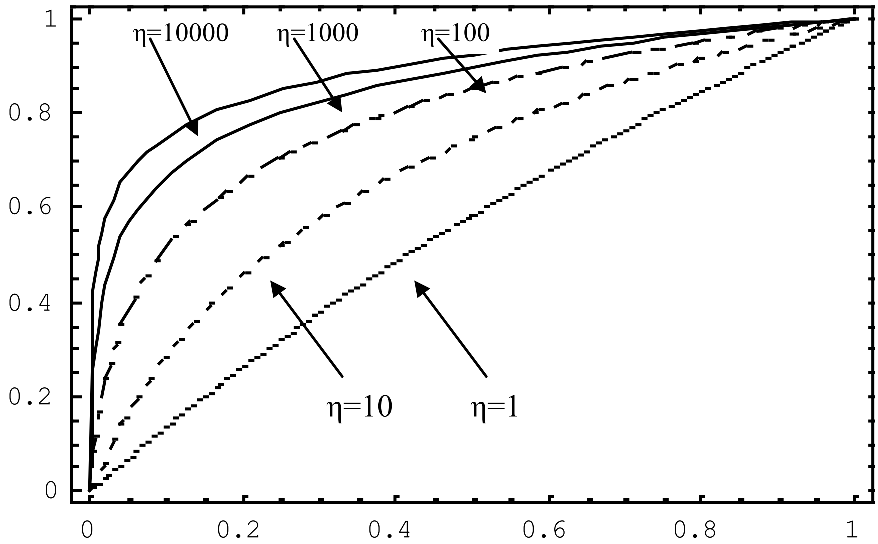

Fig. 3: The exact solution *u* of Problem 2 with different η constants.

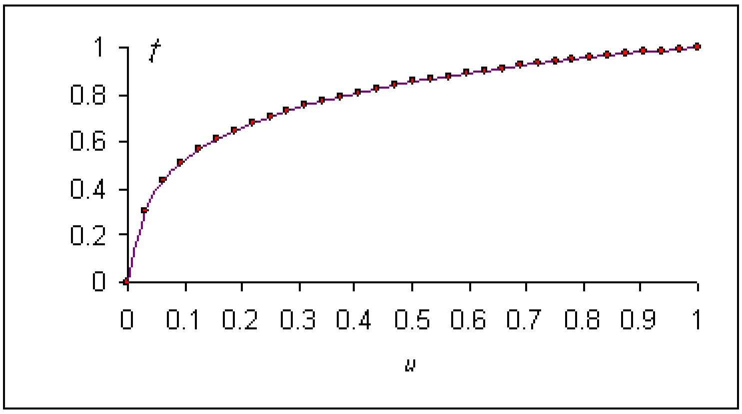

Fig. 4: The exact solution *u* of Problem 2 and the approximate solution computed by our method for, η=100, with *N*=32.

Problem 3. Next, we consider the boundary problem [13]:

$$u''(t) = \frac{2t}{1+t^2}u'(t) - \frac{2}{1+t^2}u(t) + 1,$$

$u(0)=1.25$, $u(4)=-0.95$, $t \in (0, 4)$,

with the analytic solution:

$$u(t) = 1.25 + 0.4860896526t - 2.25t^2 + 2t\arctan(t) - \frac{1}{2}\ln(1+t^2) + \frac{1}{2}t^2\ln(1+t^2).$$

Table 4 shows the comparisons of absolute errors of the QSC methods with the Runge Kutta method of the order four.

Problem 4. Finally, we consider the boundary problem [11]:

$$u''(t) = \frac{2}{t^2}u(t) - \frac{1}{t},$$

$u(2)=u(3)=0,\ t\in(2, 3)$,
with the analytic solution:

$$u(t)=\frac{1}{38}(19t-5t^2-\frac{36}{t}).$$

Table 5 appears the comparisons of absolute errors of the QSC methods with the Numerov method of order four.

Table 2. The absolute error norm for the approximate solution of Problem 1.

N	Quadratic splin collocation methods [3]		**Presented QSC Methods**	
	q=3, p=1.5	q=7, p=0.5	c_1=0.5, c_2=0.7, q=3	c_1=0.25, c_2=0.7, q=7
16	--	--	**1.735E-18**	**6.8192E-09**
32	1.49-7	1.03-5	**1.166E-18**	**2.0848E-10**
64	9.37-9	6.41-7	**6.234E-19**	**6.4440E-12**
128	5.87-10	3.74-8	**1.626E-19**	**2.0033E-13**

Table 3. The absolute error norm for the approximate solution of Problem 2.

N	Quadratic splin collocation methods[3]	**Presented QSC Methods**
	$\lvert E(w_i)\rvert$	c_1=0.25, c_2=0.7
	η=1	
16	2.26E-06	**2.0098E-10**
32	6.01E-08	**6.4137E-12**
64	2.84E-09	**2.0224E-13**
128	1.64E-10	**6.3698E-15**
	η=100	
16	4.19E-03	**7.6373E-06**, c_1=0.25, c_2=0.700074
32	2.36E-04	**3.3573E-06**, c_1=0.25, c_2=0.7027
64	1.43E-05	**5.2456E-07**, c_1=0.25, c_2=0.705611
128	8.91E-07	**1.6313E-08,** c_1=0.25, c_2=0.70752
	η=10000	
16		**4.1799E-05**, c_1=0.4131158 , c_2=0.96
32	1.62E-03	**4.0435E-05**, c_1=0.41614811, c_2=0.96
64	1.14E-04	**3.6550E-05**, c_1=0.4174018 , c_2=0.95
128	7.43E-06	**2.1389E-06** c_1=0.38706657, c_2= 0.85,
256	5.12E-07	**1.6591E-07**, c_1=0.3925753 , c_2= 0.85,

Table 4. The absolute errors for the approximate solution of Problem 3.

T	Runge-Kutta method of order four [3]		**Presented QSC Methods** c_1=0.25, c_2=0.7,	
	Absolute Error, *h*=0.2	Absolute Error, *h*=0.1	Absolute Error, *h*=0.2	Absolute Error, *h*=0.1
0.2	4.2E-05	2.0E-06	**1.57436E-08**	**7.00537E-11**
0.4	7.9E-05	5.0E-06	**2.52923E-08**	**1.38371E-10**
0.6	1.10E-04	6.0E-06	**3.01861E-08**	**1.82378E-10**
0.8	1.36E-04	8.0E-06	**3.30960E-08**	**2.06604E-10**
1.0	1.58E-04	1.0E-05	**3.53135E-08**	**2.22929E-10**
1.6	1.98E-04	1.0E-05	**3.95995E-08**	**2.62724E-10**
2.0	2.03E-04	1.2E-05	**3.98059E-08**	**2.79501E-10**
2.4	1.93E-04	1.2E-05	**3.74028E-08**	**2.84141E-10**
2.8	1.68E-04	1.1E-05	**3.22421E-08**	**2.75365E-10**
3.2	1.26E-04	7.0E-06	**2.42706E-08**	**2.52789E-10**
3.6	7.10E-05	5.0E-06	**1.34699E-08**	**2.16305E-10**
4.0	1.00E-07	1.0E-08	**1.65880E-10**	**1.65880E-10**

Table 5. The absolute errors for the approximate solution of Problem 4.

t	Numerov method of order four, *h*=1/4 [11]	**Presented QSC Methods** c_1=0.24, c_2=0.7, *h*=1/4
2.25	2.5E-06	**2.405279E-09**
2.50	2.4E-06	**2.652336E-09**
2.75	1.6E-06	**1.683887E-09**

6. Conclusions and Recommendations

A collocation approach which produces a family of order five methods has been described for the approximate solution of second-order two point boundary value problem in ordinary differential equations. Four test examples have been solved to compare the accuracy of the methods with other methods. A look at Tables 2,3,4,5 clearly shows that the presented methods are better in accuracy than other methods.

Finally, we recommend the following:

- Establishing the QSC methods for solving higher order linear and nonlinear boundary value problems in ordinary differential equations.
- Studying the QSC methods for solving boundary value problems of delay and algebraic differential equations.
- Investigating the QSC methods for numerical treatment of boundary value problems in partial differential equations.
- Applying the methods for solving problems in dynamical systems.
- Using the methods for solving problems of stiff differential equations.
- Investigating the spline collocation methods for solving boundary value problems in Volterra integro-differential equations.

REFERENCES

1. ARCHER, D., *An O(h^4) cubic spline collocation method for quasilinear parabolic equations*, SIAM J., Numer. Anal., 14(4), 1977, 620-637.
2. BURDEN, R. L. and J. D. Faires, *Numerical Analysis*, PWS-KENT, Boston Massachusetts, 1993, 559-587.
3. CHRISTARA, C. C. and K. S. Ng, *Optimal quadratic spline collocation on non-uniform partitions*, M.Sc. Thesis, Department of computer science, University of Toronto, Toronto, Ontario, Canada, 2004.
4. CHRISTARA, C. C., *Quadratic spline collocation methods for elliptic partial differential equations*, BIT, 34, 1994, 33-61.
5. DANIEL, J. W. and B. K. Swartz, *Extrapolated collocation for two-point boundary-value problems using cubic splines*, J. Inst. Maths. Applics., 16, 1975, 161-174.
6. FYFE, F. N., *The use of cubic splines in the solution of two-point boundary value problems*, Comput. J., 12, 1969. pp.188-192.
7. HALL, G. and J. M. Watt, *Modern numerical methods for ordinary differential equations*, Claredon Press, Oxford,1976.
8. HOUSTIS, E. N.; C. C. Christara, and J. R. Rice, *Quadratic-spline collocation methods for two-point boundary value problems*, International J. for Numerical Methods in Engineering, 26, 1988, 935-952.
9. HOUSTIS, E. N.; E. A. Vavalis, and J. R. Rice, *Convergence of an* O(h^4) *cubic spline collocation method for elliptic partial differential equations*, SIAM J. Numer. Anal., 25(1), 1988, 54-74.
10. IRODOTOU-ELLINA, M. and E. N. Houstis, *An* O(h^6) *quintic spline collocation method for fourth order two-point boundary value problems*, BIT, 28, 1988, 288-301.
11. JAIN, M. K., *Numerical solution of differential equations*, Wiley Eastern Limited, 1991, 173-250.
12. MAHMOUD, S. M., *A class of three-point spline collocation methods for solving delay-differential equations*, Tishreen Uni. J.-Basic Scie. Series, Vol.(28), No.(1), 2006, 163-178.
13. MATHEWS, J. H., *Numerical Methods for mathematics*, science and engineering, Prentice-Hall, Englewood Cliffs, N.J., 1992, 484-491.
14. ZHU, Y., *Optimal quartic spline collocation method for fourth order two-point boundary value problems*, M.Sc. Thesis, Department of computer science, University of Toronto, Toronto, Ontario, Canada, 2001.