مجلة جامعة تشرين للدراسات والبحوث العلمية _ سلسلة العلوم الأساسية المجلد (23) العدد (10) 2001

Tishreen University Journal for Studies and Scientific Research-Basic Science Series Vol (23) No (10) 2001

A Weak Monotone Separation Axiom

Dr.Adnan AL-Bsoul*

Alia AL-Nuaimat**

(Accepted 5/5/1999)

□ ABSTRACT □

In this paper we shall introduce a new concept of monotone sepration axioms which we shall call it monotone T1 space. We shall prove that the class of monotone T1 spaces is greater (bigger) than the class of monotone normal spaces. We obtained some results concerning the properties of this concept, in fact we proved that monotonically T1 is hereditary productive and topological property also we will prove that every monotone T1 space is T3 .this paper included open problems.

***Assosiate prof,department of math , AL albayt university , mafraq , Jordan.**

*****Master of math , faculty of arts and sciences ,department of math , AL albayt university , mafraq , Jordan.**

مجلة جامعة تشرين للدراسات والبحوث العلمية _ سلسلة العلوم الأساسية المجلد (23) العدد (10) 2001

Tishreen University Journal for Studies and Scientific Research-Basic Science Series Vol (23) No (10) 2001

مسلمة فصل رتيبة ضعيفة

الدكتور عدنان البصول*

عليا النعيمات**

(قبل للنشر في 1999/5/5)

□ الملخّص □

في هذا البحث سوف نستحدث مصطلح جديد من مسلمات الفصل الرتيبة والذي سوف نسميه فضاءات T1 الرتيبة سوف نبرهن بأن صف فضاءات T1 الرتيبة أكبر من صف الفضاءات العادية الرتيبة. حصلنا على بعض النتائج حول خصائص هذا المصطلح. في الحقيقة برهنا أن خاصية T1 الرتيبة هي وراثية ، جدائية وخاصة تبولوجية ، وأيضا سوف نبرهن أن كل فضاء T1 رتيب هو T3 . هذا لابحث يحتوي على مسائل مفتوحة .

*أستاذ مساعد ، كلية الآداب والعلوم ، جامعة آل البيت ، المفرق ، الأردن

1. Introduction. One of the most important concepts in the study of generalized metric spaces is monotone normality. C. R. Borges introduced the concept of monotone normality in 1966 as an unnamed property of all stratifiable spaces in his paper (Borges, 1966). In 1970, P. Zenor gave the property its name while announcing (Zenor, 1970) several theorems that hold in such spaces. Of special note were analogs, for monotonically normal spaces, of Katetov's metrization theorem for compacta with hereditarily normal cubes (Katetov, 1948) and of Dugundgi's extension theorem for metric spaces (Dugundgi, 1951). In 1973, the various announced results of Zenor and of Heath and Lutzer comprised the first major paper (Heath, Lutzer, and Zenor, 1973) on monotone normality. That paper is a fairly thorough investigation into the position of monotone normality in the class of generalized metric spaces. In the same year, Borges gave some additional characterizations of monotone normality (Borges, 1973). In 1996, R. C. Buck introduced the concept of monotone T_2 - spaces (Buck, 1996).

In this paper we shall introduce a concept weaker than monotone T_2 which we shall call it monotone T_1 and we shall give some results concerning this concept.

The relative topology (subspace topology) on the set A inherited by τ will be denoted by τ_A. $\Re$ denote the set of all real numbers. The τ-closure of the subset A of the topological space (X,τ) is denoted by cl(A), and the closure of the subset M of the subspace topology space (A, τ_A) is denoted by $cl_A(M)$. Let τ_{dis}, τ_{cof} denote the discrete and the cofinite topologies on a set X, respectively.

Definition 1. 1 (Buck, 1996). A T_1 space X is called *monotonically normal* if there is a function G which assigns to each ordered pair (H, K) of disjoint closed subsets of X an open set G(H, K) such that:

i) $H \subseteq G(H, K) \subseteq cl(G(H, K) \subseteq X \setminus K$;

ii) if (H', K') is a pair of disjoint closed sets such that $H \subseteq H'$ and $K \supseteq K'$, then $G(H, K) \subseteq G(H', K')$.

The function G is called a *monotone normality operator* for X.

Definition 1. 2 (Buck, 1996). A topological space X is called *monotonically T_2* if there is a function $g : X \times X \rightarrow \tau_X$ assingning to each ordered pair (x, y) of distinct points in X an open neighborhood, g(x, y), of x such that:

i) $g(x, y) \cap g(y, x) = \phi$;

ii) if $x \in cl(\cup \{g(y, x) \mid y \in M\})$, then $x \in cl(M)$.

The function g is called a *monotone T_2 operator* for X.

It is easy to see that every monotone normal space is monotone T_2.

If we return to condition (i) in Definition 1. 2 and weaken it in the following way : "for every two distinct points x, and y of X, then $x \in g(x, y)$, $y \in g(y, x)$, $y \notin g(x, y)$ and $x \notin g(y, x)$". We obtain a new definition which we call it monotone T_1.

Definition 1. 3. A topological space X is called *monotonically T_1* if there is a function $g : X \times X \rightarrow \tau_X$ assigning to each ordered pair (x, y) of distinct points in X an open neighborhood, g(x, y), of x such that:

i) for every two distinct points x, and y of X, then $x \in g(x, y)$, $y \in g(y, x)$, $y \notin g(x,y)$ and $x \notin g(y, x)$;

ii) if $x \in cl(\cup \{g(y, x) \mid y \in M\})$, then $x \in cl(M)$.

We shall call the function g is a *monotone T_1 operator* for X.

Now, we shall give two examples, the first one is a monotone T_1, and the second is not monotone T_1.

Example 1. 4. Consider the space $(\Re, \tau_{dis})$. It is easy to see that, this space is monotone T_1. In fact, define the monotone T_1 operator $g : \Re \times \Re \rightarrow \tau_{dis}$ as follows:

$$g(x, y) = \{x\}, \text{ for every pair of distinct points } x, y \text{ of } \Re.$$

Example 1. 5. Consider the space $(\Re, \tau_{cof})$. If $(\Re, \tau_{cof})$ is monotone T_1, then there exists a monotone T_1 operator $g : \Re \times \Re \rightarrow \tau_{cof}$. Let $x \neq y$ in $\Re$, so g(x, y) is an open neighborhood of x in $(\Re, \tau_{cof})$ with $y \notin g(x, y)$. Thus, we may assume that $g(x, y) = \Re \setminus \{y,$

$x_1, \ldots, x_n\}$ for some finitely many points $x_1, \ldots, x_n$ different from x. Hence, $\Re = cl(\Re \setminus \{y, x_1, \ldots, x_n\}) = cl(g(x, y))$.

Take the subset $M = \{0\}$, so, $1 \in \Re = cl(g(0,1))$,but $1 \notin \{0\} = cl(\{0\}) = cl(M)$. Hence, there is no such monotone T_1 operator g. Therefore, $(\Re, \tau_{cof})$ is not monotone T_1.

2. Some Results Concerning Monotone T_1 - Spaces. In this section we shall give the main properties of monotone T_1 - spaces. In fact, we shall prove that monotonically T_1 is hereditary, topological property and productive. Moreover we shall prove that every monotone T_1 - space is regular which is strengthen the result of Buck (Buck, 1996), every monotone T_2 - space is regular. Let us start with the following obvious results.

Theorem 2. 1. *Every monotone normal space is monotone T_1.* □

Theorem 2. 2. *Every monotone T_2 - space is monotone T_1.* □

Because of previous theorem we have the following open problem

Question 2. 3. *Is every monotone T_1 - space a monotone T_2 ?*

Theorem 2. 4 (Buck, 1996). *If a topological space X is a regular, first countable, T_1 - space, then X is monotonically T_2.*

Corollary 2. 5. *If a topological space X is a regular, first countable, T_1 - space, then X is monotonically T_1.* □

In the next result we shall prove that monotonically T_1 is hereditary.

Theorem 2. 6. *Monotonically T_1 is hereditary.*

Proof. Assume that X is monotonically T_1, and A is a subset of X. Since X is monotonically T_1, there exists a monotonically T_1 operator $g : X \times X \to \tau_X$ satisfying conditions (i), and (ii) of Definition 1. 3. Define the function $h : A \times A \to \tau_A$ in the natural way as follows:

$$h(a, b) = g(a, b) \cap A,$$

for every pair (a, b) of distinct points of A. Since g(a, b) is open in X, so $g(a, b) \cap A$ is open in the subspace topology (A, τ_A). Hence, h is well defined for every $(a, b) \in A \times A$ with $a \neq b$.

Let (a, b) be a pair of distinct points in A, so (a, b) is a pair of distinct points in X. Since g is a monotone T_1 operator, so g(a, b) is an open neighborhood of X containing the point a and does not contain the point b, also g(b, a) is an open neighborhood of X containing the point b and does not contain the point a. Thus, $h(a,b) = g(a, b) \cap A$ is an open neighborhood of A containing the point a and does not contain the point b, and $h(b, a) = g(b, a) \cap A$ is an open neighborhood of A containing the point b and does not contain the point a. Hence, condition (i) of Definition 1. 3 follows.

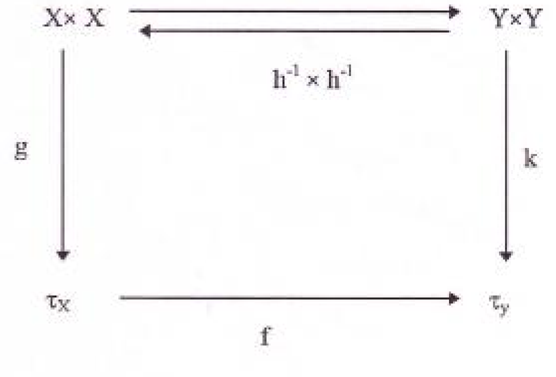
Now, assume that M is a subset of A and x is an element of A, but x is not an element of $cl_A(M)$. Then we have $cl_A(M) = H \cap A$, where H is closed in X, so x is not an element of the closed set H. Thus, we have

$x \notin cl_X(\cup \{g(y, x) \mid y \in H\}) \supseteq cl_X(\cup\{h(y, x) \mid y \in M\}) \supseteq cl_A(\cup\{h(y, x) \mid y \in M\})$.

Therefore, A is a monotonically T_1 - subspace. □

Theorem 2. 7. *Monotone T_1 is a topological property.*

Proof. Let X and Y be homeomorphic topological spaces by the homeomorphism $h : X \to Y$. Without loss of generality, assume that X is a monotone T_1 space. So, there exists a monotone T_1 operator $g : X \times X \to \tau_x$. Define the function $k : Y \times Y \to \tau_Y$ which makes the following diagram commutative



that is, $k(y_1, y_2) = f(g(h^{-1}(y_1), h^{-1}(y_2)))$ for each ordered pair (y_1, y_2) of distinct points in Y, where $f : \tau_X \to \tau_Y$ is the function defined by $f(U) = h(U)$ for each open set U in X.

Let y_1, y_2 be two distinct points in Y. Since h is bijective, so $h^{-1}(y_1)$ and $h^{-1}(y_2)$ are distinct points in X, and since X is monotone T_1, so $g(h^{-1}(y_1), h^{-1}(y_2))$ is an open neighborhood of $h^{-1}(y_1)$ in X and does not contain $h^{-1}(y_2)$, also, $g(h^{-1}(y_2), h^{-1}(y_1))$ is an open neighborhood of $h^{-1}(y_2)$ in X and does not contain $h^{-1}(y_1)$. Thus, $y_1 \in k(y_1, y_2)$, because;

$$y_1 = hh^{-1}(y_1) \in h(g(h^{-1}(y_1), h^{-1}(y_2))) = f(g(h^{-1}(y_1), h^{-1}(y_2))) = k(y_1, y_2),$$

similarly, $y_2 \in k(y_2, y_1)$, also, $y_1 \notin k(y_2, y_1)$, because;

$y_1 = hh^{-1}(y_1) \notin h(g(h^{-1}(y_2), h^{-1}(y_1))) = f(g(h^{-1}(y_2), h^{-1}(y_1)) = k(y_2, y_1)$, similarly, $y_2 \notin k(y_1, y_2)$.

So part (i) of definition 1. 3 follows.

Let M be any subset of Y, and y is a point in $cl_Y(\cup\{k(z, y) \mid z \in M\})$. So, $h^{-1}(M)$ is a subset of X and

$$\begin{aligned} h^{-1}(y) &\in h^{-1}(cl_Y(\cup\{k(z, y) \mid z \in M\})) \\ &= cl_X(h^{-1}(\cup\{k(z, y) \mid z \in M\})) \quad \text{since h is a homeomorphism} \\ &= cl_X(\cup\{h^{-1}(k(z, y)\} \mid z \in M\})) \\ &= cl_X(\cup\{h^{-1}(f(g(h^{-1}(z), h^{-1}(y))) \mid z \in M\}) \\ &= cl_X(\cup\{h^{-1}(h(g(h^{-1}(z), h^{-1}(y))) \mid z \in M\}) \\ &= cl_X(\cup\{g(h^{-1}(z), h^{-1}(y))) \mid z \in M\}), \end{aligned}$$

since g is monotone T_1 operator, so $h^{-1}(y) \in cl_X(h^{-1}(M))$, but $h^{-1}(cl_Y(M)) = cl_X(h^{-1}(M))$ and so $y = h(h^{-1}(y)) \in h(h^{-1}(cl_Y(M))) = cl_Y(M)$.

Therefore, Y is Monotone T_1 space. □

To study productivity of monotone T_1 we need the following definition.

Definition 2. 8 (Buck, 1996). Suppose $\{X_\alpha \mid \alpha \in \Lambda\}$ is a family of topological spaces. A set of the form $\Pi_{\alpha\in\Lambda} G_\alpha$ is called an *open box* if G_α is open in X_α for each α belonging to Λ. The *box topology* on $\Pi_{\alpha\in\Lambda} X_\alpha$ is the topology generated by the base of all open boxes. We designate the space $\Pi_{\alpha\in\Lambda} X_\alpha$ with the box topology by $\square_{\alpha\in\Lambda} X_\alpha$.

Theorem 2. 9. *If the product Topology on $\Pi_{\alpha\in\Lambda} X_\alpha$ is monotone T_1 - space, then X_α is monotone T_1 - space for each $\alpha \in \Lambda$.*

Proof. For each $\alpha \in \Lambda$, pick the point p_α in X_α. It is easy to show that X_β is homeomorphic to $X_\beta \times \Pi_{\alpha\in\Lambda\setminus\{\beta\}}\{p_\alpha\}$. Since $\Pi_{\alpha\in\Lambda} X_\alpha$ is monotone T_1 - space, by Theorem 2.7, X_β is a monotone T_1 space for each $\beta \in \Lambda$. $\square$

It is easy to prove the following result.

Corollary 2. 10. *If the box topology on $\square_{\alpha\in\Lambda} X_\alpha$ is monotone T_1 - space, then X_α is monotone T_1 for each $\alpha \in \Lambda$.* $\square$

Theorem 2. 11. *If X_α is a topological space which is monotonically T_1 for each $\alpha \in \Lambda$, then $\square_{\alpha\in\Lambda} X_\alpha$ is monotonically T_1.*

Proof. For $\alpha\in\Lambda$, let g_α be a monotone T_1 operator on X_α, and let $X = \Pi_{\alpha\in\Lambda} X_\alpha$. Let p and x be distinct points in X. Define the monotone T_1 operator on X as follows:

$g(p, x) = \square_{\alpha\in\Lambda} U_\alpha$, where $U_\alpha = X_\alpha$ if $p_\alpha = x_\alpha$; and $U_\alpha = g_\alpha(p_\alpha, x_\alpha)$ if $p_\alpha \neq x_\alpha$.

If p and x are distinct points in X, then there exists $\beta \in \Lambda$ such that $p_\beta \neq x_\beta$, so $g_\beta(p_\beta,x_\beta)$ is an open neighborhood in X_β containing the point p_β and does not contain the point p_β, also $g_\beta(x_\beta,p_\beta)$ is an open neighborhood in X_β containing the point x_β and does not contain the point p_β. So, g(p, x) containing p and does not contain x, and g(x,p) containing x and does not contain p. Thus part (i) of the Definition 1.3, is satisfied.

Now, suppose that $M \subset X$ and $p \notin cl(M)$. Then, there exists a basic open set U $= \square_{\alpha\in\Lambda} U_\alpha$ in X with $p \in U$ such that $U \cap M = \emptyset$. Let

$$V = \square_{\alpha\in\Lambda} U_\alpha \setminus [\, cl\,(\cup \{ g_\alpha(x_\alpha, p_\alpha) \mid x_\alpha \in X_\alpha \setminus U_\alpha \})\,].$$

Since $p_\alpha \in U_\alpha$, so $p_\alpha \notin X_\alpha \setminus U_\alpha = cl(X_\alpha \setminus U_\alpha)$ and since X_α is monotone T_1, so

$$p_\alpha \notin cl\,(\cup \{ g_\alpha(x_\alpha, p_\alpha) \mid x_\alpha \in X_\alpha \setminus U_\alpha \}).$$

Thus, $p \in V \subseteq U$, with V open in X. Suppose that, there exists an element $z \in V \cap \{\cup \{g(x, p) \mid x \in M\}\}$. Then $z \in g(y, p)$ for some $y \in M$, so $z_\alpha \in g_\alpha(y_\alpha, p_\alpha)$ for all $\alpha \in \Lambda$. However, $z \in V$, so $z_\alpha \notin \cup \{ g_\alpha(x_\alpha, p_\alpha) \mid x_\alpha \in X_\alpha \setminus U_\alpha \})$ for all $\alpha \in \Lambda$.

Therefore, for all $\alpha \in \Lambda$, $y_\alpha \notin X_\alpha \setminus U_\alpha$. Thus, for all $\alpha \in \Lambda$, we have $y_\alpha \in U_\alpha$ and $y \in U$, contradiction. Hence, $z \notin \cup \{ g(x, p) \mid x \in M\}$ and $V \cap [\cup \{g(x, p) \mid x \in M\}] = \varnothing$.

Since $p \notin cl[\cup \{ g(x, p) \mid x \in M\}]$, so condition (ii) of Definition 1. 3 is satisfied. Thus, g is a monotone T_1 operator. Therefore X is monotone T_1. □

Corollary 2. 12. *$\Box_{\alpha\in\Lambda} X_\alpha$ is monotonically T_1 iff X_α is monotonically T_1 for each $\alpha \in \Lambda$.* □

Let us close this paper by the following result..

Theorem 2. 13. *Every monotone T_1 - space is a T_3 - space.*

Proof. Let X be a monotone T_1 - space. So, there exists a monotone T_1 operator $g : X \times X \to \tau$.

Let M be a closed subset of X and x be a point not in M. Thus $x \notin cl(\cup \{g(y, x) \mid y \in M\})$, by (ii) of Definition 3. 1. Thus, $X \setminus cl(\cup \{g(y, x) \mid y \in M\})$ is an open set containing x, and since $y \in g(y, x)$ for all $y \in M$, so $M \subseteq \cup \{g(y, x) \mid y \in M\}$.

Hence, the sets $U = \cup \{g(y, x) \mid y \in M\}$ and $V = X \setminus cl(\cup \{g(y, x) \mid y \in M\})$ are disjoint open sets in X containing M and x respectively.

Thus, X is regular. Since X is T_1, therefore X is a T_3 - space. □

Since every monotone T_1 - space is a T_3 - space, so one may ask the following question.

Question 2. 14. *Is every monotone T_1 - space is a T_4 - space?*

Buck, R. E. 1996 - Some weaker monotone separation and basis properties. Top. App., Vol. 69, 1-12.

Dugundgi, J. 1951 - An extension of Tietze's theorem. Pacific J. Math., Vol. 1, 353 - 357.

Heath, R.W., Lutzer, D.J., and Zenor, P.L. 1973 - Monotonically normal spaces. Trans. Amer. Math. Soc., Vol. 178, 481-493.

Katetov, M. 1948 - Complete normality of Cartesian products. Fund. Math., Vol. 35, 271 - 174.

Zenor, P. 1970 - Monotonically normal spaces. Notices Amer. Math Soc., Vol. 17, 1034, Abstract No. 679 - G2.