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Study of Graph Partitions Approach satisfies Vizing's Conjecture

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□ ABSTRACT □

For a graph G(V,E) , a subset of vertices D is a dominating set if for each vertex x ∈V either $x \in D$, or x is adjacent to at least one vertex of D . The domination number , $\gamma(G)$, is the order of smallest dominating set of G . In [7], Vizing conjectured that $\gamma(G \times H) \geq \gamma(G) \times \gamma(H)$ for any two graphs G and H , where G×H denotes their Cartesian product . This conjecture is still open .

In this paper , we investigate following relations, if a graph H has a D-partition then it also has a K-partition, and if H has a K-partition , then Vizing's conjecture is satisfied for any graph G , after that, every cycle C_n ,$n \geq 3$, has a K-partition. Moreover, if H has a K-partition , then H satisfies the following relations $\gamma(H) \leq 2$, $P_2(H) = \gamma(H)$ and H is a perfect-dominated graph .

Key Words : Domination , Dominating Sets , Graph Products , Vizing's Conjecture.

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دراسة طريقة تجزيء للبيان تحقق تخمين فيزنج

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☐ ملخّص ☐

ليكن G(V,E) بيان ، ولتكن D مجموعة جزئية من مجموعة رؤوس البيان G، يقال عن D إنها مجموعة سيطرة للبيان G إذا كان لأجل كل رأس $x \in V$ إما $x \in D$، أو x مجاور لرأس واحد على الأقل لـ D . ونسمي عدد عناصر أصغر مجموعة سيطرة للبيان G بعدد السيطرة ونرمز له بـ $\gamma(G)$.

لقد خمن فيزنج في [7] إن المتراجحة $\gamma(G \times H) \geq \gamma(G) \times \gamma(H)$ محققة من أجل أي بيانين G و H، يعني أن عدد السيطرة للجداء الديكارتي لأي بيانين G×H أكبر أو يساوي حاصل جداء عدد السيطرة للبيان G بعدد السيطرة للبيان H . ومن الجدير بالذكر أن هذا التخمين ما يزال مفتوحاً حتى الآن.

تم في هذا العمل مناقشة وإثبات القضايا الآتية ، إذا كان للبيان H D–تجزئة يكون له K–تجزئة، وإذا كان له K–تجزئة ، فإن تخمين فيزنج يتحقق من أجل أي بيان G، كما تم إثبات أن كل حلقة C_n، $n \geq 3$، تملك K–تجزئة.

أخيراً تم إثبات أنه إذا كان للبيان H K–تجزئة فإنه يحقق القضايا الآتية: $\gamma(H) \leq 2$، و $P_2(H) = \gamma(H)$ وأن البيان H يكون تام السيطرة.

الكلمات المفتاحية: السيطرة، مجموعات السيطرة، جداء البيان، تخمين فيزنج.

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Introduction

Let G= (V,G) be finite , undirected , simple graph . A subset D of vertices of G is called dominating if every vertex $y \in V - D$ is adjacent to some vertex x $\in D$. The domination number of G is $\gamma(G) = min\{|D| : \text{D is dominating set of G}\}$.

If G and H are graphs , then their Cartesian product , denoted by G× H , is the graph with vertex set V(G×H)=V(G)×V(H) and edge set:

$E(\text{G} \times \text{H}) = \{(x_1,\text{y}_1)(x_2,\text{y}_2): x_1 = x_2 \text{ and } \text{y}_1\text{y}_2 \in \text{E(H) or } x_1x_2 \in \text{E(G)and } \text{y}_1 = \text{y}_2\}$.

Let G be a graph , we denote by N(x) and N[x] the open and the closed neighborhoods of a vertex x, respectively . Also let N(A)= $U_{x\in A}N(x) and\ N[x] =_{x\in A} N(x)$ be the open and closed neighborhoods of a set $A \subseteq V$.

In 1963 , Vizing presented the following conjecture :

Conjecturing Vizing in[7] : If G and H are two graphs then $\gamma(G \times H) \geq \gamma(G)\,\gamma(H)$.

Let the class of graphs VC be defined by

$G \in VC\ if\ and\ only\ if\ \gamma(G \times H) \geq \gamma(G) \times \gamma(H)\ for\ any\ graph\ H.$

One way to prove that if a graph $G \in VC$ then certain partition conditions of this graph are satisfied.

So, Baracalkin and German in [1] proved that $G \in VC$ provided that G contains a C-partition, i.e. a partition of V(G) into $\gamma(G)$ subsets each of which induces a complete subgraph.

The concept of a CC-partition was introduced by Faudree , Schelp , and Shreve [2] . A partition of V(G) into $\gamma(G)$ subsets , is a CC-partition if for any set $A \subseteq V$ such that $|A| < \gamma(G)$ there is no partite set V_i with $V_i \cap A = \emptyset$ and $V_i \subseteq N(A)$. Again $G \in VC$ whenever it contains a CC-partition .

Chen , Piotrowski and Shreve in [4] , introduced the concept of extracted partition . Let $v = \{V_1, V_2, \dots, V_k\}$ be a partition of V=V(G) . Say , that the partite set V_i is covered by a set $A \subseteq V$ if either $V_i \cap A \neq \emptyset$ or $V_i \subseteq N(A)$. Let $d_A = |\{V_i \in v : V_i\ \ is\ covered\ by\ A\}|$. A partition $v = \{V_1, V_2, \dots, V_k\}$ is called extracted if $d_A \leq |A|$ for any set $A \subseteq V$. The extraction number $\chi(G)$ is the largest order of an extracted partition of V(G) . The authors in [4] proved the following inequalities:

$P_2(G) \leq \chi(G) \leq \gamma(G)\ and\ \gamma(G \times H) \geq \chi(G) \times \gamma(H)$

In particular , if $\chi(G) = \gamma(G)$, then $G \in VC$, also , they established the following results:

$\gamma(G \times H) \geq \gamma(G)P_2(H) + P_2(G)\big(\gamma(H) - P_2(H)\big)$.

Also in [3] , El-Zahar and Pareek proved that every G , with $\gamma(G) = m$ has a partition $V(G) = V_1 \cup \dots \cup V_m$, where each V_i is a dominating set for $\bar{G}$, the complement of G . They employed this fact to show that if $\gamma(G) = 2$ then $G \in VC$.

In [8], Maheswari et al. study direct product graphs of Cayley graphs with arithmetic graphs and present matching dominating set of these graphs.

In [9], Hedetniemi finds unique minimum dominating sets in Cartesian product graphs, after that, studies an external graph theory problem and determines the maximum number of edges in a graph having a unique minimum independent dominating set or a unique minimum maximal irredundant set of two cardinality.

The above results suggest that dealing with Vizing's conjecture through graph partitions are fruitful . We introduce the following definitions .

Definition 1 : Let H be a graph with $\gamma(H) = m$, a partition $V(H) = V_1 \cup \ldots \cup V_m$ such that for each i=1,2,…,m there exists $y_i \in V_i$ with $d(y_i, V_j) \geq 2$ $for\ j \neq i$ is called a two-distance partition or simply a D-partition.

Definition 2 : Let H a graph with $\gamma(H) = m,$ a partition of the vertices set of H , $V(H) = V_1 \cup \ldots \cup V_m$ is called a K-partition whenever the graph H` , obtained from H by adding edges in such a way that each V_i becomes a complete subgraph , still satisfies $\gamma(H') = m$.

We investigate these graph partitions in the next section . In particular, we prove that a graph G satisfies $G \in VC$ whenever it has a D-partition or has a K-partition. We also present classes of graphs which satisfy these partitions . Before presenting these results, we recall some results which will be needed later .

A set B is a 2-packing of graph G , if $d(x, y) \geq 3$ for any $x, y \in B$ two vertices, the 2-packing number, $P_2(G)$, is the order of a largest 2-packing set of G.

In [5], Theorem (A) is presented by Meir & Moon:

(a) For any graph G , $P_2(G) \leq \gamma(G)$.

(b) For any tree T , $P_2(T) = \gamma(T)$.

Importance of Research and its objectives:

It is well known that the graph theorem is primarily due to their great usefulness in applications. Moreover, finding the dominating set and the domination number of graphs are very important in practical and scientific applications, the numerical solutions of purposed problems are very important. So, they contribute in solving several problems.

This paper aims to find the partition conditions which imply that a graph G satisfies Vizing's conjecture a dominating set, and the domination number of graph.

Methodology

Research methods are directly depended on graph theorem techniques and computer algorithms. Moreover, some classical theorems in linear algebra are very useful for satisfying purposed techniques. Algorithms and programming languages can be used for obtaining numerical results. Finally, the various references cited at the end of this paper are observed .

Results and Discussion

Theorem 1 : A graph H which has a D-partition it also has a K-partition.

Proof : Suppose that H has a D-partition , $V(H) = V_1 \cup \ldots \cup V_m$, where $m = \gamma(H)$. Thus each V_i ,$i = 1, \ldots, m,$ contains a vertex x_i such that $d(x_i, V_j) \geq 2$ for each $j \neq i.$ Consider now that the graph H' obtained from H by adding edges in such a way that each V_i becomes a complete subgraph, obviously $\gamma(H') \leq \gamma(H).$ Consider any dominating set D of H' , if for some i , $1 \leq i \leq m$, we have $D \cap V_i = \emptyset$ then x_i is not dominated, thus $|D| \geq m,$ this implied that $\gamma(H') = m.$ Therefore, $V(H) = V_1 \cup \ldots \cup V_m$ is also a K-partition of H.

This completes the proof of the Theorem 1.

Remark : The converse of Theorem 1 is not true . A cycle C_n for n≡1 , 2(mod 3) has a K-partition but it does not have a D-partition .

Theorem 2 : If a graph H has a K-partition , then this graph H ∈ VC .

Proof : Let H have a K-partition $V(H) = V_1 \cup \ldots \cup V_m$, where $m = \gamma(H)$. We denote by H' the graph obtained from H by adding edges to each V_i such that it becomes a complete subgraph . Thus, $\gamma(H') = \gamma(H) = m$. We have a partition $V(H') = V'_1 \cup \ldots \cup V'_m$ of H' where each V'_i induces a complete subgraph. By the result of Baracalkin and German [1] , we have

$\gamma(G \times H') \geq \gamma(G)\gamma(H')$ for any graph G .

The graph G × H' contains G×H as a spanning subgraph which implies that $\gamma(G \times H) \geq \gamma(G \times H')$. Thus, we deduce that

$\gamma(G \times H) \geq \gamma(G \times H') \geq \gamma(G)\gamma(H') = \gamma(G)\gamma(H)$.

It yields that $G \in VC$. This completes the proof of the Theorem2 .

Theorem 3 : Every cycle C_n ,$n \geq 3$, has a K-partition.

Proof : Let C_n denote the cycle $x_1, x_2, \ldots, x_n$, then $\gamma(C_n) = [n/3]$. Let n = 3k + r , $0 \leq r < 3$ and write $\gamma(C_n) = m$ where m=k if r=0 and m=k+1 otherwise.

We define a partition $V_1 \cup \ldots \cup V_m$ as follows

Case 1 , r=0

Let $V_i = \{x_{3i-2}, x_{3i-1}, x_{3i}\}$; $i = 1,2, \ldots, k$.

Case 2, r=1

Let $V_i = \{x_{3i-1}, x_{3i}\}$; $i = 1,2, \ldots, k$,

$V_{k+1} = \{x_1, x_4, \ldots, x_{n-3}, x_n\}$

Case 3 . r= 2

Let $V_i = \{x_{3i-1}, x_{3i}\}$; $i = 1,2, \ldots, k$

$V_{k+1} = \{x_1, x_4, \ldots, x_{n-4}, x_{n-1}, x_n\}$.

It is not difficult to check that , in each case $V_1 \cup \ldots \cup V_m$ is a K-partition of C_n . The proof of the Theorem 3 is completed.

Let us remark that if n ≡ 0(mod 3) , then the above the K-partition of C_n is also a D-partition . Note that , in a D-partition , a Partite set must contain some vertex and all its neighbors . Hence , a partite set in a D-partition of C_n must contain at least three vertices . Therefore, a cycle C_n, n≡1 or 2(mod 3) , is an example of a graph that has a K-partition but hasn't a D-partition .

According to [6] , a set of vertices S of a graph H is called a perfect dominating set if every vertex of H is either in S or is adjacent to exactly one vertex of S . H is called a perfect- dominated graph if it has a perfect dominating set S with $|S| = \gamma(H)$.

Theorem 4 : If a graph H has a K-partition , then H satisfies each one of the following relations:

1) $\gamma(H) \leq 2$
2) $P_2(H) = \gamma(H)$
3) H is a perfect-dominated graph .

Proof : 1) $\gamma(H) \leq 2$

1) The result is trivial when $\gamma(H) = 1$. Let $\gamma(H) = 2$, according to [3], there is a partition of $V(H) = V_1 \cup V_2$ such that each V_i is a dominating set for H , i =1,2. Let H' be obtained from H by adding edges to each V_i so that it becomes a complete subgraph.

Suppose that $X \in V_i\ (i = 1\ or\ 2)$. Then there is a vertex $y \epsilon V_{3-i}$ which dominates x in H. Thus, $xy \notin E(H)$. This implies that $\gamma(\mathrm{H}') = 2$. Therefore, $V(H) = V_1 \cup V_2$, is a K-partition of H.

2) $P_2(H) = \gamma(H)$

Let H be a graph with $P_2(H) = \gamma(H) = m$, and suppose that $B = \{y_1, y_2, \dots, y_m\}$ is a 2-packing of H. We define a partition $V(H) = V_1 \cup V_2 \cup \dots \cup V_m$ as follows. for each vertex $x \in H$, choose j such that $d(x, y_j) = min\{d(x, y_j) : 1 \leq i \leq m\}$ and let $x \in V_j$. This defines the required partition. Now, we have $N[y_i] \subseteq V_i$ for each i=1,2,…,m . Consequently, $V_1 \cup V_2 \cup \dots \cup V_m$ is indeed a D-partition of H.

3) H is a perfect-dominated graph.

Let $S = \{y_1, y_2, \dots, y_m\}$ be a perfect dominating set of H with $|S| = \gamma(H)$. Let $V_i = N[y_i]$, i =1,2,…,m , then $V(H) = V_1 \cup V_2 \cup \dots \cup V_m$ is a D-partition of H. Consequently, the proof of the theorem 4 is completed.

Conclusions

In this study, the graph partitions were successfully investigated, if a graph H has a D-partition then it has also a K-partition, either if H has a K-partition , then H $\in$ VC , every cycle C_n ,$n \geq 3$, has a K-partition, moreover, if H has a K-partition , then H satisfies the following relations:

1) $\gamma(H) \leq 2$

2) $P_2(H) = \gamma(H)$

3) H is a perfect-dominated graph .

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