AFuzzy Dual Ideals and Their Lattice Structure in Bck-Algebras

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□ ABSTRACT □

In this paper, we investigate several structural properties of fuzzy dual ideals within the framework of BCK-algebras. This result provides a direct algebraic characterization that links duality with the classical notion of filtering in BCK-structures.

Furthermore, we examine prime anti-dual fuzzy ideals and show that any such fuzzy ideal satisfies a functional identity of the form

$$\mu\big((x \vee y) \wedge z\big) = \mu(x \wedge z) \vee \mu(y \wedge z)$$

which reveals how prime anti-dual behavior interacts with meet and join operations.

Finally, we characterize fuzzy dual ideals through the identity

$$\mu(x \wedge y) = \mu(x) \wedge \mu(y) \quad \forall x, y \in X$$

which offers a functional criterion for determining when a fuzzy set in X behaves as a fuzzy dual ideal. These results collectively deepen the understanding of fuzzy dual structures in BCK-algebras and provide new tools for analyzing their order-theoretic behavior.

Keywords: BCK Algebra, ideal, fuzzy ideal, fuzzy dual ideal, prime fuzzy dual ideal, anti-fuzzy dual ideal.

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المثاليات الثنوية الضبابية وبنيتها الشبكية في جبر BCK

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ملخّص

في هذا البحث ندرس عدة خصائص بنيوية للمثاليات الثنوية الضبابية في جبر BCK. يقدم هذا العمل توصيفاً جبرياً مباشراً يربط مفهوم الثنوية بالمفهوم الكلاسيكي للمرشحات في بنى BCK. علاوة على ذلك نقوم بدراسة المثاليات الضبابية الثنوية المضادة الأولية ونبين أن كل مثال من هذا النوع يحقق هوية دالية من الشكل المعطى

$$\mu\big((x \vee y) \wedge z\big) = \mu(x \wedge z) \vee \mu(y \wedge z)$$

مما يكشف عن كيفية تفاعل سلوك الثنوية المضادة الأولية مع عمليات الاجتماع والتقاطع.

وأخيراً نقدم توصيفاً للمثاليات الضبابية من خلال هوية دالية معينة والتي توفر معيار لتحديد متى تكون مجموعة ضبابية في X مثالية ثنوية ضبابية

$$\mu(x \wedge y) = \mu(x) \wedge \mu(y) \quad \forall x, y \in X$$

كما تسهم هذه النتائج في تعميق فهم البنى الثنوية الضبابية في جبر BCK وتزود أدوات جديدة لتحليل سلوكها الترتيبي

الكلمات المفتاحية: جبر BCK ، مثالية، مثالية ضبابية، مثالية ضبابية ثنوية، مثالية ضبابية ثنوية أولية، مثالية ثنوية ضبابية مضادة.

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Introduction:

The study of BCK-algebra was initiated by Imai and Iséki [1] as a generalization of the concept of set-theoretic difference and propositional calculi. The concept of fuzzy sets was introduced by L. A. Zadeh [2], and Rosenfeld [3] was the first to apply fuzzy set theory to algebraic systems in 1971. Later, O. Xi extended the use of fuzzy sets to BCK-algebras, opening the way for a broad line of research on fuzzy algebraic structures and their logical interpretations.

Following these foundational developments, several fuzzy extensions of classical ideals in BCK-algebras have been introduced and studied, including fuzzy ideals, fuzzy filters, anti-fuzzy ideals, and dual fuzzy ideals. These fuzzy generalizations have proven to be powerful tools for analyzing uncertainty, graded membership, and structural behaviors within algebraic systems. Among them, fuzzy dual ideals represent an important class that captures duality phenomena under the algebraic operations of BCK-structures.

Despite the increasing attention given to fuzzy concepts in BCK-algebras, the order-theoretic nature and lattice behavior of fuzzy dual ideals remain less explored. Motivated by this gap, the present work aims to investigate new characterizations and structural properties of fuzzy dual ideals, with particular emphasis on their lattice organization and their interaction with prime anti-dual fuzzy ideals. The results obtained here extend and unify several strands of research within the theory of fuzzy BCK-algebras.

Preliminaries:

Definition 2.1. [4] Let $(X,*,0)$ be a groupoid with a distinguished element 0 and a binary operation $*$. Then $(X,*,0)$ is a BCI-algebra if: for all $x,y,z \in X$,

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0 \; and \; y * x = 0 \; imply \; x = y$

In a BCI-algebra X, a partially ordered relation ≤ can be deIned by

$x \leq y$ If and only if $x * y = 0$

A BCI-algebra is said to be a BCK-algebra if it satisfies:

(IIV) $0 * x = 0$ for all $x \in X$.

A BCK/BCI-algebra X has the following properties: [5]

(b1) $(\forall x \in X)\,(x * 0 = x)$.

(b2) $(\forall x,y,z \in X)\,((x * y) * z = (x * z) * y)$.

(b3) $(\forall x,y,z \in X)\,(x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x)$.

(b4) $(\forall x,y,z \in X)\,((x * z) * (y * z) \leq x * y)$.

In particular, if X is a BCK-algebra then the following property hold:

(b5) $(\forall x,y \in X)\,((x * y) * x = 0)$.

Definition 2.2. [6] A BCK-algebra X is said to be bounded if there exists an element $1 \in X$ such that $x \leq 1$ for all $x \in X$.

For elements x and y of a bounded BCK-algebra X, we denote [7]

$$x \vee y = N(Nx \wedge Ny) \; where \; Nx = 1 * \mathrm{x}$$

Definition 2.3. [8] A BCK-algebra X is called commutative if

$$x * (x * y) = y * (y * x) \text{ For all } x,y \in X \text{ where } x \wedge y = y * (y * x)$$

Definition 2.4. [9] A BCK algebra X is called implicative if

$$x * (y * x) = x \text{ for all } x,y \in X$$

Definition 2.5. [10] A BCK-algebra X is positive implicative if it satisfies the following condition:

$$(\forall x, y, z \in X)((x * z) * (y * z) = (x * y) * z).$$

Definition 2.6. [4] A nonempty subset I of a BCK/BCI-algebra X is said to be an ideal if it satisfies:

(I1) $0 \in I$, and (I2) for all $x, y \in X$; $x * y \in I$ and $y \in I$ imply $x \in I$.

Any ideal I has the property: (I3) $x \in I$ and $y \leq x$ imply y ∈I

Definition 2.7. [11] An ideal of commutative BCK-algebras X is said to be prime if $x \wedge y \in I$ implies $x \in I$ or $y \in I$.

Definition 2.8. [12] A nonempty subset D in a BCK-algebra X is said to be a dual ideal of X if it satisfies:

(D1) $1 \in D$

(D2) $N(Nx * Ny) \in D$ and $y \in D$ imply $x \in D$.

Definition 2.9. [13] Let X be a set. A fuzzy set μ in X is a function

$$\mu : \mathrm{X} \rightarrow [0, 1].$$

And the complement of μ, denoted by μ^c is the fuzzy subset of X given by

$\mu^c(x) = 1 - \mu(x)$ for all $x \in X$

Definition 2.10. [13] A fuzzy set μ in a BCI-algebra X is said to be a fuzzy ideal of X if it satisfies:

(F1) $\mu(0) \geq \mu(x)$,

(F2) $\mu(x) \geq min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

If μ is fuzzy ideal in X then, for $x, y, z \in X$, such that $(x * y) * z = 0$, then

$$\mu(x) \geq min\{\mu(y), \mu(z)\}$$

Definition 2.11. [14] A fuzzy subset μ of a BCK-algebra X is called an anti-fuzzy ideal of X if

(i) $\mu(0) \leq \mu(x)$,

(ii) $\mu(x) \leq max\{\mu(x * y), \mu(y)\}$,

For all $x, y \in X$.

Definition 2.12. [12] Let X be a BCK-algebra. A fuzzy subset μ of X is said to be a fuzzy dual ideal of X if it satisfies:

(i) μ (1) ≥ μ (x) for all x in X;

(ii) $\mu(x) \geq min\{\mu(N(Nx * Ny), \mu(y)\}$ For all x, y in X.

Definition 2.13. [15] A non-constant fuzzy ideal μ of a commutative BCK-algebra X is said to be prime if

$$\mu\,(x \wedge y) \leq \mu(x) \vee \mu(y) \ \ for\ all\ x, y\ of\ X$$

Definition 2.14. [6] Let μ be a non-constant fuzzy dual ideal of X. then μ is said to be prime if

$$\mu\,(x \vee y) \leq \mu(x) \vee \mu(y) \ \ for\ all\ x, y\ of\ X$$

Definition 2.15. [6] Let μ be a fuzzy subset of X. then μ is said to be an anti-fuzzy dual ideal of X if

(i) $\mu\,(1) \leq \mu\,(x), \forall\, x \in X$

(ii) $\mu\,(x) \leq \max\{\mu\,(N(Nx * Ny)), \mu\,(y)\}\ \forall\, x, y \in X$

Definition 2.16. [6] Let μ be an anti-fuzzy dual ideal of X. Then μ is said to be prime if

$$\mu\,(x \vee y) \geq \mu(x) \wedge \mu(y) \ \ for\ all\ x, y\ of\ X$$

3. Results

Theorem 3.1. Let X be implicative and finite BCK algebra, and let μ be prime anti dual ideal then

$$\mu\big((x \vee y) \wedge z\big) = \mu\,(x \wedge z)\ V\ \mu\,(y \wedge z)$$

proof. We have $(x \vee y) \wedge z = (x \wedge z) V (y \wedge z)$, because

$$\big((x \vee y) \wedge z\big) * \big((x \wedge z) V (y \wedge z)\big) = (N(Ny * x) \wedge z) * ((x * Nz) V (y * Nz)$$
$$= (N(Ny * x) * Nz) * N\big(N(x * Nz) * (y * Nz)\big)$$
$$= (z * (Ny * x) * N(N(x * Nz) * (y * Nz))$$
$$= \big(N(x * Nz) * (y * Nz)\big) * N\big(z * (Ny * x)\big)$$
$$= \big(N(x * Nz)\big) * N\big(z * (Ny * x)\big) * (y * Nz)$$
$$= \Big(\big(z * (Ny * x)\big) * (x * Nz)\Big) * (y * Nz)$$
$$= \big((N(Ny * x) * Nz) * (x * Nz)\big) * (y * Nz)$$
$$= \big((N(Ny * x) * x) * Nz\big) * (y * Nz)$$
$$= \big((N(Ny * x) * Nz) * x\big) * (y * Nz)$$
$$= \Big(\big(z * (Ny * x)\big) * x\Big) * (y * Nz)$$
$$= (\big(z * x) * (Ny * x)\big) * (y * Nz)$$
$$= \big((z * Ny) * x\big) * (y * Nz)$$
$$= \big((z * Ny) * (y * Nz)\big) * x$$
$$= \big((z * Ny) * (z * Ny)\big) * x$$
$$= 0 * x = 0$$

So $(x V y) \wedge z \leq (x \wedge z) V (y \wedge z)$

And we have, $x \wedge z \leq x$ & $y \wedge z \leq y$ so $(x \wedge z) V (y \wedge z) \leq x V y$

Also $x \wedge z \leq z$ & $y \wedge z \leq z$ so $(x \wedge z) V (y \wedge z) \leq z$

Then $(x \wedge z) V (y \wedge z) \leq (x V y) \wedge z$

So $(x V y) \wedge z = (x \wedge z) V (y \wedge z)$, and $\mu((x V y) \wedge z) = \mu((x \wedge z) V (y \wedge z))$

$$= \mu(x \wedge z) V \mu(y \wedge z)$$

Theorem 3.2. Let X be finite BCK algebra, then the family of fuzzy dual ideals represent lattice.

Proof. let A represent the family of fuzzy dual ideals in X. And

$\forall f, g \in A$; $(f V g)(x) = f(x) V g(x)$

$(f \wedge g)(x) = f(x) \wedge g(x)$

so to complete the proof we have to prove that, $f V g \in A$ & $f \wedge g \in A$

$(f V g)(1) = f(1) V g(1) \geq f(x) V g(x) = (f V g)(x)$

$(f V g)(x) = f(x) V g(x) \geq f\big(N(Nx * Ny) \wedge f(y)\big) V (g(N(Nx * Ny) \wedge g(y))$

$$= \Big(f\big(N(Nx * Ny)\big) \vee g\big(N(Nx * Ny)\big)\Big) \wedge \big(f(y) V g(y)\big)$$
$$= (f V g)\big(N(Nx * Ny)\big) \wedge (fV g)(y)$$

That means $f V g$ is fuzzy dual ideal in X, so $f V g \in A$

Also $(f \wedge g)(1) = f(1) \wedge g(1) \geq f(x) \wedge g(x) = (f \wedge g)(1)$

$(f \wedge g)(x) = f(x) \wedge g(x) \geq f\big(N(Nx * Ny) \wedge f(y)\big) \wedge (g(N(Nx * Ny) \wedge g(y))$

$$= \Big(f\big(N(Nx * Ny)\big) \wedge g\big(N(Nx * Ny)\big)\Big) \wedge \big(f(y) \wedge g(y)\big)$$
$$= (f \wedge g)\big(N(Nx * Ny)\big) \wedge (f \wedge g)(y)$$

So $f \wedge g \in A$.

Theorem 3.3. Let X be implicative and finite BCK algebra, and let μ is fuzzy set in X so μ is fuzzy dual ideal iff

$$\mu(x \wedge y) = \mu(x) \wedge \mu(y) \quad \forall x, y \in X$$

proof. suppose that μ is fuzzy dual ideal in X. we have $\mu(x) \geq \mu\left(N(y*x)\right) \wedge \mu(y)$

We replace x with $x \wedge y$ so get, $\mu(x \wedge y) \geq \mu\left(N\left(y*(x \wedge y)\right)\right) \wedge \mu(y)$

$$= \mu\left(N\left(y*(y*(y*x))\right)\right) \wedge \mu(y)$$

$$= \mu\left(N(y*x)\right) \wedge \mu(y)$$

But $N(y*x) \geq x \Rightarrow \mu(N(y*x)) \geq \mu(x) \Rightarrow \mu(N(y*x)) \wedge \mu(y) \geq \mu(x) \wedge \mu(y)$

$$\Rightarrow \mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$$

We have $x \wedge y \leq x \;\&\; x \wedge y \leq y \Rightarrow \mu(x \wedge y) \leq \mu(x) \;\&\; \mu(x \wedge y) \leq \mu(y)$

$$\Rightarrow \mu(x \wedge y) \leq \mu(x) \wedge \mu(y)$$

So, $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$.

Let $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$ replace x with $N(y*x)$ so get,

$\mu\left(N(y*x)\wedge y\right) = \mu\left(N(y*x)\right) \wedge \mu(y)$

But $N(y*x) \wedge y = y*\left(y*N(y*x)\right)$

$$= y*\left((y*x)*N_y\right)$$

$$= y*\left((y*N_y)*x\right)$$

$$= y*(y*x) = x \wedge y$$

So, $\mu(x \wedge y) = \mu\left(N(y*x)\right) \wedge \mu(y)$

But $\mu(x \wedge y) = \mu(x) \wedge \mu(y) \leq \mu(x)$ so, $\mu(x) \geq \mu\left(N(y*x)\right) \wedge \mu(y)$

and the first condition from fuzzy dual ideal satisfied.

Also we have $x \leq 1$ and $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$

we replace y with 1 we find $\mu(x \wedge 1) = \mu(x) \wedge \mu(1)$

$\Rightarrow \mu(x) = \mu(x) \wedge \mu(1) \Rightarrow \mu(1) \geq \mu(x)$

So, we conclude that μ is fuzzy dual ideals in X.

References:

[1] Y. Imai and K. Iski, On Axiom Systems of Propositional Calculi. XIV, Comm. by Kinjir, Kunugi, M.J.A., Jan. 12, 1966.

[2] L. A. Zadeh, Fuzzy Sets, Information And Control 8, 338--353 (1965).

[3] A. Rosenfeld, Fuzzy Groups, Journal Of Mathex4tical Analysis and Applications 35, 512-517 (1971).

[4] J. Meng and X. Guo, on fuzzy ideals in BCK/BCI-algebras, fuzzy sets and systems, 149 509-525, (2005).

[5] Y. Jun, K. Ja Lee, Positive implicative vague ideals in BCK-algebras, Annals of Fuzzy Mathematics and Informatics Vol 1, No. 1, pp. 97- 105, (January 2011).

[6] M.M. Zahedi, G.H.A. Bozorgee, Some Equivalent Conditions Of Fuzzy Prime Dual Ideals In BCK-Algebra, Scientiae Mathematical, Vol.2. No. 3, 361-371,(1999).

[7] A. Najafi and A. B. Saeid, Solvable BCK-Algebras, Chankaya University Journal of Science and Engineering Volume 11, No. 2 19–28,(2014).

[8] H.A.S. Abujabal, M. Aslam and A.B. Thaheem, A Representation Of Bounded Commutative BCK-Algebras, Internat. J. Math. & Math. Sci. VOL. 19 NO. 4 733-736,(1996).

[9] J.Ahsan, E.Y.Deeba, A.B.Thahee, On Prime Ideal And Associate Spectrum Of BCK – Algebras, Islmabad University. Pakistan,1989.

[10] Y. Jun, E. Roh, and M. Öztürk, Positive Implicative Ideals of BCK-Algebras Based on

Intuitionistic Falling Shadows, Mathematics, 6, 149, 2018.
[11] S. Abdullah, Intuitionistic fuzzy prime ideals of BCK-algebras, Annals of Fuzzy Mathematics and Informatics Volume x, No. x, (Month 201y), pp. 1- xx (2013).
[12] J. Meng and Y. B. Jun, Fuzzy Dual Ideals in BCK-Algebras, Comm. Korean Math. Soc. 8, No.2, pp. 22&-231,(1993).
[13] A. Kordi and A.Moussavi, On fuzzy ideals of BCI-algebras, PU.M.A.Vol. 18, No . 3-4, pp. 301-310, (2007).
[14] S.M. Hong and Y.B. Jun ,Anti fuzzy ideals in BCK-algebras , Kyungpook Math .J. 38, 145-150,(1998).
[15] Y. B. Jun, E. H. Roh, J. Meng and X. L. Xin, Fuzzy Prime and Fuzzy Irreducible Ideals in BCK-Algebras, Soochow Journal Of Mathematics Volume 21 No. 1, pp. 49-56, January 1995.