Metric Spaces Over Partially Ordered Rings

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□ ABSTRACT □

In this paper, we present a systematic study of a new class of generalized metric spaces, called POR-metric spaces, in which the distance function takes its values in a partially ordered ring satisfying specific algebraic compatibility conditions that ensure coherence between the algebraic structure and the order relation. This work falls within current directions in the generalization of metric spaces, which seek to move beyond the traditional numerical framework of distance by replacing real-valued distances with ordered or algebraic structures, thereby combining additional algebraic information into the topological structure.

The significance of this approach lies in enabling a deeper interaction between topological concepts and algebraic structures, as well as providing greater flexibility in notions of convergence, continuity, and topology generation. In this context, we establish the fundamental properties of POR-metric spaces, and we define and study POR-balls, open and closed sets, interior, exterior, and boundary points, as well as certain separation axioms, highlighting similarities and differences with the classical metric case. Overall, POR-spaces constitute a coherent extension of existing concepts and provide a flexible and extensible framework for further research in topology, fixed point theory, and abstract analysis.

Key words: PO-rings, POR-metric, POR-open ball, POR-open set, POR-closed set.

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الفضاءات المترية فوق الحلقات المرتبة جزئياً

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☐ ملخّص ☐

في هذه المقالة، نقدّم دراسة منهجية لفئة جديدة من الفضاءات المترية المعمّمة، تُسمّى فضاءات مترية من النمط POR، حيث تأخذ دالة المسافة قيَمها في حلقة مرتّبة جزئياً تحقق شروط توافق جبرية محدّدة تضمن انسجام البنية الجبرية مع علاقة الترتيب. ويندرج هذا العمل ضمن الاتجاهات الحديثة في تعميم مفهوم الفضاء المتري، والتي سعت إلى تجاوز الإطار العددي التقليدي للمسافة، من خلال استبدال القيم الحقيقية ببنى مرتّبة أو جبرية أوسع، بما يسمح بإدماج معلومات جبرية إضافية داخل البنية التبولوجية ذاتها.

تتبع أهمية هذا التوجه من كونه يتيح تفاعلاً أعمق بين المفاهيم التبولوجية والبنى الجبرية، ويمنح مرونة أكبر في صياغة مفاهيم التقارب والاستمرارية وتوليد التبولوجيا. وفي هذا السياق، نثبت الخصائص الأساسية للفضاءات المترية من النمط POR ، ونعرّف وندرس الكرات من النمط POR ، والمجموعات المفتوحة والمغلقة، والنقاط الداخلية والخارجية والحدودية، إضافة إلى بعض بديهيات الفصل، مع إبراز أوجه التشابه والاختلاف مقارنةً بالحالة المترية الكلاسيكية. وبصورة عامة، تمثل الفضاءات من النمط POR امتداداً منسجماً للمفاهيم القائمة، وتوفّر إطاراً مرناً وقابلاً للتطوير لمزيد من الدراسات في التبولوجيا ونظرية النقاط الثابتة والتحليل المجرد.

الكلمات المفتاحية: حلقة مرتبة جزئياً، تابع مسافة من النمط POR، كرة مفتوحة من النمط POR، مجموعة مفتوحة من النمط POR، مجموعة مغلقة من النمط POR.

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Significance and Objectives of the Study:

The significance of this study lies in introducing a new class of generalized metric spaces in which the distance function takes values in a partially ordered ring with suitable compatibility conditions. This framework extends several known generalizations of classical metric spaces and emphasizes the interaction between order and algebraic structures. The main objective is to define POR-metric spaces and investigate their basic properties, including the induced topological notions such as open and closed sets and balls. The results provide a flexible setting for further developments in topology and analysis.

Research Methods and Resources:

The research falls within the field of theoretical mathematics, specifically in the theory of spaces. Accordingly, the methods adopted in this work are purely theoretical and rely primarily on fundamental concepts from set theory, topological spaces, and algebraic ideals.

Some of the symbols and terminology used in this paper:

(X, d) POR-metric space. $B(a, r)$ POR-open ball. $\bar{B}(a, r)$ POR-closed ball. $Int_{POR}(A)$ POR-interior of A. $Cl_{POR}(A)$ POR-closure points of A. $\acute{A}_{POR}$ POR-limit points of A. $Is_{POR}(A)$ POR-isolated points of A.

Introduction:

The concept of a metric space is fundamental in analysis and topology, as it provides a unified framework for studying convergence, continuity, and various qualitative properties of functions and spaces. Since the classical definition introduced by Fréchet in his seminal work [1], numerous generalizations of metric spaces have been proposed, mainly motivated by the need to extend analytical tools to broader contexts and more flexible structures.

Early attempts to relax the classical axioms of metrics include semi-metric and quasi-metric spaces, as studied by Freudenthal [2] and Wilson [3], where symmetry or the triangle inequality is weakened. These developments paved the way for a systematic investigation of generalized metric structures and their associated topologies.

A prominent direction of research has focused on modifying the codomain of the distance function, replacing the real numbers by more general ordered or algebraic structures. This viewpoint has been extensively discussed in the literature; see, for instance, the survey by Kabil and Pouzet [4], which highlights the deep connections between generalized metrics, ordered sets, and graph-based structures. Further evidence of the flexibility of this approach is provided by Kopperman [5], who showed that every topology can be generated by a suitable generalized metric.

More recently, attention has been directed toward metrics taking values in ordered structures. In this context, poset-valued distances have been investigated systematically, as illustrated in the theory of poset metrics developed by Firer et al. [6]. Such approaches emphasize the role of order-theoretic properties in shaping metric-like notions and their applications. In addition, other recent generalizations have been proposed in different frameworks, such as neutrosophic metric spaces introduced by Smarandache and Ali [7], further illustrating the diversity of metric-type structures.

Within this general framework, partially ordered rings offer a particularly flexible setting. They combine algebraic operations with a partial order that may be required to be compatible with addition, allowing the construction of distance functions that respect both algebraic and order structures. This makes partially ordered rings a natural candidate for defining generalized metrics that extend classical notions while retaining meaningful topological behavior.

Motivated by these considerations, this paper introduces and investigates POR-metric spaces, where the distance function is a stable mapping taking values in a partially ordered ring satisfying suitable conditions. The proposed framework unifies several existing generalizations of metric spaces and provides a setting in which topological properties induced by ring-valued distance functions can be studied in a systematic way.

1.Preliminaries:

Definition 1.1 [1]: For any non-empty set X, the function $d: X \times X \to \mathbb{R}^+$ is called a metric if it satisfied the following:

C1. $d(x,x) = 0$

C2. $d(x,y) = 0 \Longrightarrow x = y$

C3. $d(x,y) = d(y,x)$

C4. $d(x,y) + d(y,z) \geq d(x,z)$

And (X,d) is called a metric space.

Definition 1.2 [3]: If $d: X \times X \to \mathbb{R}^+$ satisfied (C1) and (C4) then (X,d) is called a quasimetric space.

Definition 1.3 [2]: If $d: X \times X \to \mathbb{R}^+$ satisfied (C1), (C2) and (C3) then (X,d) is called a semi-metric space.

Definition 1.4 [5]: If $d: X \times X \to \mathbb{R}^+$ satisfied (C1), (C3) and (C4) then (X,d) is called a pseudometric.

Definition 1.5 [8]: A partially ordered set (poset) is a set equipped with a binary relation $(\leq)$ that is reflexive, antisymmetric, and transitive.

Definition 1.6 [9]: Let A be a poset with the relation $(\leq)$, and $x, y \in A$, we say that:

1. (x,y) has a supremum element (denoted by $sup(x,y)$) if:

$\begin{cases} x \leq sup(x,y) \\ y \leq sup(x,y) \end{cases}$. And for any $t \in R$; $\begin{cases} x \leq t \\ y \leq t \end{cases}$ we get: $t \geq sup(x,y)$ where $sup(x,y) \in A$.

2. (x,y) has an infimum element (denoted by $inf(x,y)$) if:

$\begin{cases} x \geq inf(x,y) \\ y \geq inf(x,y) \end{cases}$. And for any $t \in R$; $\begin{cases} x \geq t \\ y \geq t \end{cases}$ we get: $t \leq inf(x,y)$ where $inf(x,y) \in A$.

Definition 1.7 [9]: A poset A with the relation $(\leq)$ is called a Lattice if for any $x, y \in A$ there exist $\sup(x,y), \inf(x,y) \in A$.

2. Partially ordered rings

Definition 2.1: Let $(R, +, .)$ be a ring, assume that $(\leq)$ is a partial order relation defined on R. We say that R is a partially ordered ring (PO-ring in short) if and only if for all $a, b, c, d \in R$ the following conditions hold:

1. If $a \leq b$, $c \leq d$ then $a + c \leq b + d$
2. $a \leq b \Leftrightarrow -a \geq -b$

Corollary 2.1: Let R be a PO-ring and let $a, b \in R$ such that $a \leq b$ then:

$$a + c \leq b + c \text{ and } a - c \leq b - c \text{ for all } c \in R$$

Proof: we have $a \leq b\ and\ c \leq c$ since R is partially ordered, then by condition (1) of Definition 2.1 we obtain: $a + c \leq b + c$
On the other hand, we have:
$-c \leq -c$ and $a \leq b$ so as before we deduce:

$$a - c \leq b - c$$

Example2.1: The ring of neutrosophic real numbers is defined as follows:

$$\mathbb{R}(I) = \{a + bI\ ; a, b\ \in \mathbb{R}\ , I^2 = I\ \}$$

Define addition, multiplication and the partial order relation $(\leq)$ on $\mathbb{R}(I)$ as follows:
For all $a + bI, c + dI \in \mathbb{R}(I)$:
$(a + bI) + (c + dI) = (a + c) + I(b + d)$
$(a + bI).(c + dI) = ac + I(ad + bc + bd)$
$a + bI\ \leq c + dI\ \Leftrightarrow \begin{cases} a \leq c \\ a + b \leq c + d \end{cases}$
$\mathbb{R}(I)$ is a PO-ring, since:
If $\begin{cases} a + bI\ \leq c + dI \\ m + nI\ \leq s + tI \end{cases}$; For any $a + bI, c + dI, m + nI\ , s + tI \in \mathbb{R}(I)$: then,
$\begin{cases} a \leq c \\ a + b \leq c + d \end{cases}$ and $\begin{cases} m \leq s \\ m + n \leq s + t \end{cases}$ thus, $\begin{cases} a + m \leq c + s \\ a + m + b + n \leq c + s + d + t \end{cases}$
so : $(a + m) + I(b + n) \leq (c + s) + I(d + t)$
hence, $(a + bI) + (m + nI) \leq (c + dI\ ) + (s + tI\ )$
Also: $a + bI\ \leq c + dI \Leftrightarrow \begin{cases} a \leq c \\ a + b \leq c + d \end{cases} \Leftrightarrow \begin{cases} -a \geq -c \\ -a - b \geq -c - d \end{cases} \Leftrightarrow$
which gives : $-(a + bI) \geq -(c + dI)$
Hence, the ring $\mathbb{R}(I)$ satisfies the axioms of a PO-ring
Example2.2: The ring of split-complex numbers is defined as:

$$S = \{a + bJ\ ; a, b\ \in \mathbb{R}\ , J^2 = 1\ , J \notin \mathbb{R}\ \}$$

With the following operations:
$\forall\ a + bJ, c + dJ \in S$:
$(a + bJ) + (c + dJ) = (a + c) + J(b + d)$
$(a + bJ).(c + dJ) = (ac + bd) + J(ad + bc)$
Moreover, S is a PO-ring, since it satisfies the two conditions:

i. If $\begin{cases} a + bJ\ \leq c + dJ \\ m + nJ\ \leq s + tJ \end{cases}$ then,
$\begin{cases} a - b \leq c - d \\ a + b \leq c + d \end{cases}$ and $\begin{cases} m - n \leq s - t \\ m + n \leq s + t \end{cases}$ which implies,
$\begin{cases} a + m - (b + n) \leq c + s - (d + t) \\ a + m + b + n \leq c + s + d + t \end{cases}$
so: $(a + m) + J(b + n) \leq (c + s) + J(d + t)$
hence, $(a + bJ) + (m + nJ) \leq (c + dJ\ ) + (s + tJ\ )$

ii. $a + bJ\ \leq c + dJ \Leftrightarrow \begin{cases} a - b \leq c - d \\ a + b \leq c + d \end{cases} \Leftrightarrow \begin{cases} -(a - b) \geq -(c - d) \\ -(a + b) \geq -(c + d) \end{cases} \Leftrightarrow$
$\Leftrightarrow \begin{cases} -a - (-b) \geq -c - (-d) \\ -a + (-b) \geq -c + (-d) \end{cases}$
$\Leftrightarrow -a + (-b)J \geq -c + (-d)I\ \Leftrightarrow\ -(a + bI) \geq -(c + dI)$

Definition 2.2: Let R be a PO-ring, we define :
1. $R^+ = \{a \in R\ ; a \geq 0\}$
2. $R^- = \{a \in R\ ; a \leq 0\}$
3. $R_c = R^+ \cup R^-$

Theorem 2.1: Let R be a PO-ring, then:

1. $R^+ \cap R^- = \{0\}$, $R^+ \cup R^- \neq R$ (In general)
2. $a \in R^+ \Leftrightarrow -a \in R^-$
3. R^+ is closed under (+)
4. R^- is closed under (+)

Proof:

1. $\forall a \in R^+ \cap R^- \Leftrightarrow a \in R^+ and\ a \in R^- \Leftrightarrow a \geq 0\ and\ a \leq 0 \Leftrightarrow a = 0$

Thus, $R^+ \cap R^- = \{0\}$

To show that $R^+ \cup R^- \neq R$ let's take the Example2.1:

$$\mathbb{R}(I)^+ = \{a + bI\ ; a \geq 0\ , a + b \geq 0\}\ , \mathbb{R}(I)^- = \{a + bI\ ; a \leq 0\ , a + b \leq 0\}$$

We can see that for $x = 2 - 4I \in \mathbb{R}(I), 2 \geq 0\ but\ 2 + (-4) = -2 \leq 0$, so $x \notin \mathbb{R}(I)^+, x \notin \mathbb{R}(I)^-$ with $x \in \mathbb{R}(I)$, therefore, $R^+ \cup R^- \neq R$

2. $a \in R^+ \Leftrightarrow a \geq 0 \Leftrightarrow -a \leq -0 = 0 \Leftrightarrow -a \in R^-$
3. $\forall a, b \in R^+ \Rightarrow a \geq 0\ , b \geq 0 \Rightarrow a + b \geq 0 + 0 = 0 \Rightarrow a + b \in R^+$
4. $\forall a, b \in R^- \Rightarrow a \leq 0\ , b \leq 0 \Rightarrow a + b \leq 0 + 0 = 0 \Rightarrow a + b \in R^-$

Definition 2.3: Let R be a PO-ring, and $x, y \in R$ we say that R is Lattice-PO-ring (L-PO-ring in short) if for every pair $(x, y) \in R^2$ we can define a supremum and infimum element.

Example 2.3: For the (PO-ring) $\mathbb{R}(I)$ defined in Example2.1, for $a + bI, c + dI \in \mathbb{R}(I)$ consider:

$$x = max(a, c) + I[max(a + b, c + d) - max(a, c)]$$
$$y = min(a, c) + I[min(a + b, c + d) - min(a, c)]$$

Then, $x = sup(a + bI, c + dI)\ , y = inf(a + bI, c + dI)$, that is because:

$x \geq a + bI\ , x \geq c + dI$ since: $\max(a, c) \geq a, c, and$

$max(a + b, c + d) - max(a, c) + max(a, c) = max(a + b, c + d) \geq a + b, c + d$

And for any $t = m + nI \in \mathbb{R}(I)$ with $t \geq a + bI$ and $t \geq c + dI$

so, $m \geq a\ and\ m \geq c$ so $m \geq \max(a, c)$ and:

$m + n \geq a + b\ and\ m + n \geq c + d\ so\ m + n \geq max(a + b, c + d)$

Hence, $m + nI \geq max(a, c) + I[max(a + b, c + d) - max(a, c)]$ then, $t \geq x$.

For y, it can be shown by a similar way.

As a result, $\mathbb{R}(I)$ is a L-PO-ring.

Example 2.4: For the PO-ring S, defined in Example2.2, for $a + bJ, c + dJ \in S$ consider:

$$x = x_1 + Jx_2;$$

$x_1 = \frac{1}{2}[max(a + b, c + d) + max(a - b, c - d)]$,

$x_2 = \frac{1}{2}[max(a + b, c + d) - max(a - b, c - d)]$

$$y = y_1 + Jy_2;$$

$y_1 = \frac{1}{2}[min(a + b, c + d) + min(a - b, c - d)]$,

$y_2 = \frac{1}{2}[min(a + b, c + d) - min(a - b, c - d)]$

Then, $x = sup(a + bJ, c + dJ)\ , y = inf(a + bJ, c + dJ)$, that is because:

$x \geq a + bJ\ , x \geq c + dJ$ since: $x_1 + x_2 = max(a + b, c + d) \geq a + b, c + d, and$

$x_1 - x_2 = max(a - b, c - d) \geq a - b, c - d$

And for any $t = m + nJ \in S$ with $t \geq a + bJ$ and $t \geq c + dJ$

so, $m - n \geq a - b\ and\ m - n \geq c - d$ so $m - n \geq max(a - b, c - d) = x_1 - x_2$ and:

$m + n \geq a + b\ and\ m + n \geq c + d\ so\ m + n \geq max(a + b, c + d) = x_1 + x_2$

Hence, $m + nJ \geq x_1 + x_2J$ then, $t \geq x$.

Also for y, it can be shown by a similar way.
As a result, S is a L-PO-ring.
Remark 2.1: It is not a necessary for $sup(x,y)\ or\ \ inf(x,y)$ to be equal to $x\ or\ y$
for example, take $\alpha = 2 - J\ ,\beta = 1 + J$ belong to the PO-ring defined in the Example2.2 then we can see from the previous example that:

$$sup(\alpha,\beta) = x = \frac{1}{2}(2+3) + \frac{1}{2}J(2-3) = \frac{5}{2} - \frac{1}{2}J \notin \{\alpha,\beta\}$$

$$inf(\alpha,\beta) = y = \frac{1}{2}(1+0) + \frac{1}{2}J(1-0) = \frac{1}{2} + \frac{1}{2}J \notin \{\alpha,\beta\}$$

Corollary 2.2: let R be an L-PO-ring, suppose that $(x,y) \in R^2$ with $x \le y$, then:

$$sup(x,y) = y, inf(x,y) = x$$

Proof: we have:
$x \le y, y \le y$ and for every $t \in R; x \le t, y \le t$ then $y \le t$. thus $sup(x,y) = y$.
$x \le y, x \le x$ and for every $t \in R; t \le x, t \le y$ then $t \le x$. thus $inf(x,y) = x$.
Definition 2.4: Let R be a PO-ring, we say that:

1) R has the property φ_R iff:
$$\forall\ a,b \in R; a \ge b\ then\ ac \ge bc\ for\ any\ c \in R^+$$
2) R has the property φ_L iff:
$$\forall\ a,b \in R; a \ge b\ then\ ca \ge cb\ for\ any\ c \in R^+$$
3) R has the property φ if and only if R satisfies both of φ_R and φ_L.

Corollary 2.3: Let R be a PO-ring with the property φ_R (φ_L, respectively) then for $a,b \in R\ such\ that\ a \ge b\ \ and\ \ for\ c \in R^-$ then, $ac \le bc\ (ca \le cb$, respectively)
Proof: we will prove it for φ_R, and it will be similar for φ_L:
For $c \in R^-$ we have $c \le 0$, then $-c \in R^+$, by the property φ_R:
$-ac \ge -bc$, and by the condition (2) of Definition 2.1 we deduce that $ac \le bc$.
Example 2.5: For the (PO-ring) $\mathbb{R}(I)$ defined in Example2.1,
let $a + bI, c + dI \in \mathbb{R}(I)$ such that $a + bI \ge c + dI$ then $a \ge c\ and\ a + b \ge c + d$
and let $m + nI \in \mathbb{R}(I)^+$ which implies that $m \ge 0\ and\ m + n \ge 0$
So, $am \ge cm$ … (2.1)
and $(a+b)(m+n) \ge (c+d)(m+n)$ i.e. :
$am + an + bm + bn \ge cm + cn + dm + dn$ … (2.2)
on the other hand, we know that $(a+bI)(m+nI) = am + (an + bm + bn)I$
and: $(c+dI)(m+nI) = cm + (cn + dm + dn)I$
then, by the two inequalities (2.1),(2.2) we notice that $(a+bI)(m+nI) \ge (c+dI)(m+nI)$
Therefore, the PO-ring $\mathbb{R}(I)$ has the property φ_R, and it will be similar for φ_L.
Example 2.6: Let R be the PO-ring S defined in Example2.2, using the same argument as in Example 2.5 we can show that S also has the property φ.

3. POR-metric space:

Definition 3.1: Let R be a PO-ring and X be a non-empty set with $d: X \times X \to R^+$ satisfied the following conditions for all $x,y,z \in X$

1. $d(x,y) = d(y,x)\ \ , d(x,y) = 0 \Leftrightarrow x = y$
2. $d(x,y) + d(y,z) \ge d(x,z)$

Then we call d a POR-metric with values in the PO-ring R , and (X,d) a POR-metric space over the PO-ring R.
Theorem 3.1: Every metric is a POR-metric
Proof: Let $d: X \times X \to \mathbb{R}^+$ be a metric, then for all $x,y,z \in X$ we have:

1. $d(x,y) \geq 0$
2. $d(x,y) = d(y,x)$, $d(x,y) = 0 \Leftrightarrow x = y$
3. $d(x,y) + d(y,z) \geq d(x,z)$

Note that: by the first condition the codomain of d is $\mathbb{R}^+ \subset \mathbb{R}$ a PO-ring under the usual addition and multiplication of real numbers, with the standard order. Since d satisfies the axioms of the metric, which are as same as the POR-metric axioms, it follows that d is a POR-metric with values in the PO-ring $\mathbb{R}^+$..
Hence, every metric is a POR-metric

Example 3.1: For the ring of neutrosophic real numbers is defined in the Example2.1, take $X = \mathbb{R}^2$ and define $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}(I)$ by:

$$d\big((a,b),(c,d)\big) = |a-c| + I(|b-d| - |a-c|)$$

Let us verify that d is a POR-metric with values in $\mathbb{R}(I)$
Let $(a,b),(c,d),(m,n) \in \mathbb{R}^2$, then:

1. $d\big((a,b),(c,d)\big) = |a-c| + I(|b-d| - |a-c|) =$
   $|c-a| + I(|d-b| - |c-a|) = d\big((c,d),(a,b)\big)$

Also, $d\big((a,b),(c,d)\big) = 0 \Leftrightarrow |a-c| + I(|b-d| - |a-c|) = 0 + 0I$

$$\Leftrightarrow \begin{cases} |a-c| = 0 \\ |b-d| - |a-c| = 0 \end{cases} \Leftrightarrow \begin{cases} a = c \\ b = d \end{cases} \Leftrightarrow (a,b) = (c,d)$$

2. $d\big((a,b),(c,d)\big) + d\big((c,d),(m,n)\big)$
   $= |a-c| + I(|b-d| - |a-c|) + |c-m| + I(|d-n| - |c-m|)$
   $= |a-c| + |c-m| + I(|b-d| + |d-n| - |a-c| - |c-m|)$
   On the other hand:

$d\big((a,b),(m,n)\big) = |a-m| + I(|b-n| - |a-m|)$
we can notice that:
$|a-m| = |a-c+c-m| \leq |a-c| + |c-m|$ and
$|b-n| - |a-m| + |a-m| = |b-n| = |b-d+d-n| \leq |b-d| + |d-n|$
hence, $d\big((a,b),(c,d)\big) + d\big((c,d),(m,n)\big) \geq d\big((a,b),(m,n)\big)$.

Example 3.2: For the ring split-complex numbers defined in the Example2.2, take $X = \mathbb{R}^2$ and define $d: \mathbb{R}^2 \times \mathbb{R}^2 \to S$ by:

$$d\big((a,b),(c,d)\big) = \frac{1}{2}(|b-d| + |a-c|) + \frac{1}{2}J(|b-d| - |a-c|)$$

then d is a POR-metric with values in S , and so $(\mathbb{R}^2, d)$ is a POR-metric space over S.
To prove it, let $(a,b),(c,d),(m,n) \in \mathbb{R}^2$, then:

1. $d\big((a,b),(c,d)\big) = \frac{1}{2}(|b-d| + |a-c|) + \frac{1}{2}J(|b-d| - |a-c|) =$
   $\frac{1}{2}(|d-b| + |c-a|) + \frac{1}{2}J(|d-b| - |c-a|) = d\big((c,d),(a,b)\big)$

Also, $d\big((a,b),(c,d)\big) = 0 \Leftrightarrow$
$\frac{1}{2}(|b-d| + |a-c|) + \frac{1}{2}J(|b-d| - |a-c|) = 0 + 0J$

$$\Leftrightarrow \begin{cases} |b-d| + |a-c| = 0 \\ |b-d| - |a-c| = 0 \end{cases} \Leftrightarrow \begin{cases} a = c \\ b = d \end{cases} \Leftrightarrow (a,b) = (c,d)$$

2. $d\big((a,b),(c,d)\big) + d\big((c,d),(m,n)\big)$

$$= \frac{1}{2}(|b-d| + |a-c|) + \frac{1}{2}J(|b-d| - |a-c|) + \frac{1}{2}(|d-n| + |c-m|) + \frac{1}{2}J(|d-n| - |c-m|)$$

$$= \frac{1}{2}(|b-d|+|a-c|+|d-n|+|c-m|) + \frac{1}{2}J(|b-d|+|d-n|-|a-c|-|c-m|)$$

On the other hand:

$d\big((a,b),(m,n)\big) = \frac{1}{2}(|b-n|+|a-m|) + \frac{1}{2}J(|b-n|-|a-m|)$

we can notice that:

$\frac{1}{2}(|b-n|+|a-m|) + \frac{1}{2}(|b-n|-|a-m|) = |b-n| = |b-d+d-n| \leq |b-d|+|d-n|$

$\frac{1}{2}(|b-n|+|a-m|) - \frac{1}{2}(|b-n|-|a-m|) = |a-m| = |a-c+c-m| \leq |a-c|+|c-m|$

hence, $d\big((a,b),(c,d)\big) + d\big((c,d),(m,n)\big) \geq d\big((a,b),(m,n)\big)$.

Example 3.3: Let us take $\mathbb{R}^{\mathbb{R}}$ the ring of real-valued functions with one variable over $\mathbb{R}$, equipped with the pointwise addition and pointwise multiplication, and let's define the partial order relation as:

$$for\ f,g \in \mathbb{R}^{\mathbb{R}} : f \leq g \iff f(x) \leq g(x)\ \ \forall x \in \mathbb{R}$$

It can be easily shown that $(\mathbb{R}^{\mathbb{R}},+,.)$ is a PO-ring

Define $d: \mathbb{R}^{\mathbb{R}} \times \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ by:

$$d(f,g) = |f-g|\ ; d\big(f(x),g(x)\big) = |f-g|(x) = |f(x)-g(x)|$$

Let's prove that d is a POR-metric with values in $\mathbb{R}^{\mathbb{R}}$. Let $f,g,h \in \mathbb{R}^{\mathbb{R}}$ then,

1. $\forall x \in \mathbb{R}$: $d\big(f(x),g(x)\big) = |f(x)-g(x)| = |g(x)-f(x)| = d\big(g(x),f(x)\big)$

$so, d(f,g) = d(g,f)$

$d(f,g) = 0 \Leftrightarrow |f-g|(x) = 0\ \ \forall x \in \mathbb{R} \iff f(x) = g(x) \Leftrightarrow f \equiv g$

2. $d(f,g) + d(g,h) = |f-g| + |g-h|$

   and $d(f,h) = |f-h| = |f-g+g-h| \leq |f-g| + |g-h|$

   thus, ($\mathbb{R}^{\mathbb{R}}$,d) is a POR-metric space over $\mathbb{R}^{\mathbb{R}}$.

Example 3.4: Take $M_n(\mathbb{R})$, the ring of all $n \times n$ matrices over $\mathbb{R}$, under the addition and multiplication between matrices and let's define the partial order relation as:

$$for\ A = (a_{ij}), B = (b_{ij}) \in M_n(\mathbb{R})\ :\ \ A \leq B \iff a_{ij} \leq b_{ij}\ \ for\ all\ 1 \leq i,j \leq n$$

It can be easily shown that $(M_n(\mathbb{R}),+,.)$ is a PO-ring.

Define $d: M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by:

$$d(A,B) = (d_{ij});\ \ d_{ij} = |a_{ij} - b_{ij}|\ \ \ \forall 1 \leq i,j \leq n$$

A simple argument shows that d is a POR-metric with values in $M_n(\mathbb{R})$.

Theorem 3.2: Let $(X,d_1),(Y,d_2)$ be two metric spaces, where $d_1: X \times X \to \mathbb{R}^+$and $d_2: Y \times Y \to \mathbb{R}^+$, consider the inner product $X \times Y$ and define $d: (X \times Y) \times (X \times Y) \to \mathbb{R}^+ \times \mathbb{R}^+$ as follows:

$$d\big((x_1,y_1),(x_2,y_2)\big) = \big(d_1(x_1,x_2), d_2(y_1,y_2)\big)\ for\ all\ x_1,x_2 \in X,\ \ y_1,y_2 \in Y$$

Then, $((X \times Y), d)$ is a POR-metric space over $\mathbb{R} \times \mathbb{R}$. Furthermore, the PO-ring $\mathbb{R} \times \mathbb{R}$ has the property φ.

Proof: It can be simply verified that $\mathbb{R} \times \mathbb{R}$ is a PO-ring under the usual addition, multiplication and order relation defined as follows:

$$(a,b) \leq (c,d) \Leftrightarrow a \leq c\ and\ b \leq d\ for\ all\ (a,b),(c,d) \in \mathbb{R}^+ \times \mathbb{R}^+$$

Let's prove that it has the property φ:
let $(a,b),(c,d) \in \mathbb{R} \times \mathbb{R}$ such that: $(a,b) \leq (c,d)$ this implies that: $a \leq c, b \leq d$
for any $(m,n) \in (\mathbb{R} \times \mathbb{R})^+$ then $m \geq 0, n \geq 0$. So, $am \leq cm, bn \leq dn$
hence, $(am,bn) \leq (cm,dn)$.
We now verify that d is a PO-metric with values in $\mathbb{R} \times \mathbb{R}$
for all $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in X \times Y$

1. $d\big((x_1,y_1),(x_2,y_2)\big) = \big(d_1(x_1,x_2), d_2(y_1,y_2)\big) = \big(d_1(x_2,x_1), d_2(y_2,y_1)\big)$
   $= d\big((x_2,y_2),(x_1,y_1)\big)$ that is because d_1, d_2 are metrics.

$d\big((x_1,y_1),(x_2,y_2)\big) = 0 \Leftrightarrow \big(d_1(x_1,x_2), d_2(y_1,y_2)\big) = 0 \Leftrightarrow \begin{cases} d_1(x_1,x_2) = 0 \\ d_2(y_1,y_2) = 0 \end{cases} \Leftrightarrow$
$\begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases} \Leftrightarrow (x_1,y_1) = (x_2,y_2)$

2. $d\big((x_1,y_1),(x_2,y_2)\big) + d\big((x_2,y_2),(x_3,y_3)\big) =$
   $\big(d_1(x_1,x_2), d_2(y_1,y_2)\big) + \big(d_1(x_2,x_3), d_2(y_2,y_3)\big) =$
   $\big(d_1(x_1,x_2) + d_1(x_2,x_3), d_2(y_1,y_2) + d_2(y_2,y_3)\big) \geq$
   $\big(d_1(x_1,x_3), d_2(y_1,y_3)\big) = d\big((x_1,y_1),(x_3,y_3)\big)$.

that is because d_1, d_2 are metrics.

4. Balls in POR-metric space:

Definition 4.1: let (X,d) be a POR-metric space over the PO-ring R, then:
For $a \in X$, we say that:

1. The set $B(a,r) = \{x \in X\,; d(x,a) < r\,, r \in R^+\}$ is a POR-open centered at a with radius r .
2. The set $\bar{B}(a,r) = \{x \in X\,; d(x,a) \leq r\,, r \in R^+\}$ is a POR-closed ball centered at a with radius r .
3. The set $B^c(a,r) = \{x \in X\,; d(x,a) \geq r\,, r \in R^+\}$ is the exterior of the POR-open ball $B(a,r)$.
4. The set $\bar{B}^c(a,r) = \{x \in X\,; d(x,a) > r\,, r \in R^+\}$ is the exterior of the POR-closed ball $\bar{B}(a,r)$.
5. The set $S(a,r) = \{x \in X\,; d(x,a) = r\,, r \in R^+\}$ is the POR-sphere centered at a with radius r .

Remark 4.1: If R has no definition for the relation $a < b$ instead of $a \leq b$ then $B(a,r) = \bar{B}(a,r)$ and $B^c(a,r) = \bar{B}^c(a,r)$

Example 4.1: Let's consider the POR-metric space defined in Example 3.1,
then for $(a_1,a_2) \in \mathbb{R}^2$:
$B(\,(a_1,a_2),r) = \big\{(x_1,x_2) \in \mathbb{R}^2; d\big((x_1,x_2),(a_1,a_2)\big) < r, r = r_1 + r_2 I \in \mathbb{R}(I)\big\}$
$d\big((x_1,x_2),(a_1,a_2)\big) < r_1 + r_2 I$
$\Leftrightarrow |x_1 - a_1| + I(|x_2 - a_2| - |x_1 - a_1|) < r_1 + r_2 I$
$\Leftrightarrow \begin{cases} |x_1 - a_1| < r_1 \\ |x_2 - a_2| < r_1 + r_2 \end{cases} \Leftrightarrow \begin{cases} a_1 - r_1 < x_1 < a_1 + r_1 \\ a_2 - r_1 - r_2 < x_2 < a_2 + r_1 + r_2 \end{cases}$
$\Rightarrow B(\,(a_1,a_2),r) = \left\{(x_1,x_2) \in \mathbb{R}^2; \begin{cases} a_1 - r_1 < x_1 < a_1 + r_1 \\ a_2 - r_1 - r_2 < x_2 < a_2 + r_1 + r_2 \end{cases}; r = r_1 + r_2 I \in \mathbb{R}(I)\right\}$

Hence, the POR-open ball $B\big((a_1,a_2), r_1 + r_2 I\big)$ is the open boxes in $\mathbb{R}^2$ centered at (a_1,a_2) with sides of length $2r_1$ and $2(r_1 + r_2)$, respectively.

The POR-closed ball $\bar{B}\big((a_1,a_2),r_1+r_2I\big)$ is the closed boxes in $\mathbb{R}^2$ centered at (a_1,a_2) with sides of length $2r_1$ and $2(r_1+r_2)$, respectively.
The exterior of the POR-open ball:
$$B^c\big((a_1,a_2),r\big)=\big\{(x_1,x_2)\in\mathbb{R}^2; d\big((x_1,x_2),(a_1,a_2)\big)\geq r_1+r_2I\in\mathbb{R}(I)\big\}$$
$$=\left\{(x_1,x_2)\in\mathbb{R}^2\,;\begin{cases}|x_1-a_1|\geq r_1\\ |x_2-a_2|\geq r_1+r_2\end{cases}\right\}$$
Then, the exterior of the POR-open ball $B\big((a_1,a_2),r_1+r_2I\big)$ is the region that lies outside a horizontal strip of width $2r_1$ centered at a_1, and at the same time outside a vertical strip of width $2(r_1+r_2)$ centered at a_2 including their boundaries.

The exterior of the POR-closed ball $\bar{B}\big((a_1,a_2),r_1+r_2I\big)$ is the region that lies outside a horizontal strip of width $2r_1$ centered at a_1, and at the same time outside a vertical strip of width $2(r_1+r_2)$ centered at a_2 excluding their boundaries.

The POR-spheres:
$$S(\,(a_1,a_2),r_1+r_2I)=\big\{(x_1,x_2)\in\mathbb{R}^2;\ d\big((x_1,x_2),(a_1,a_2)\big)=r_1+r_2I\big\}$$
$$\Leftrightarrow |x_1-a_1|+I(|x_2-a_2|-|x_1-a_1|)=r_1+r_2I.$$
$$\Leftrightarrow\begin{cases}|x_1-a_1|=r_1\\ |x_2-a_2|-|x_1-a_1|=r_2\end{cases}\Leftrightarrow\begin{cases}|x_1-a_1|=r_1\\ |x_2-a_2|=r_1+r_2\end{cases}\Leftrightarrow\begin{cases}x_1=a_1\mp r_1\\ x_2=a_2\mp(r_1+r_2)\end{cases}$$
The POR-spheres consist the four vertices of the same rectangle:
$\big(a_1\mp r_1,a_2\mp(r_1+r_2)\big)$.
Example 4.2: Let's consider the POR-metric space defined in Example 3.3 then for $g\in\mathbb{R}^{\mathbb{R}}$
$$B(g,r)=\{f\in\mathbb{R}^{\mathbb{R}}; d(f,g)<r\,;r\in\mathbb{R}^{\mathbb{R}}\}=\{f:\mathbb{R}\to\mathbb{R},|f-g|(x)<r(x)\forall x\in\mathbb{R}\}$$
$$\{f:\mathbb{R}\to\mathbb{R}; |f(x)-g(x)|<r(x)\ \forall x\in\mathbb{R}\}$$
$$d(f,g)<r\Leftrightarrow\ |f-g|(x)<r(x)\ \ \forall x\in\mathbb{R}\Leftrightarrow|f(x)-g(x)|<r(x)\ \ \forall x\in\mathbb{R}$$
$$\Leftrightarrow g(x)-r(x)<f(x)<g(x)+r(x)\ \forall x\in\mathbb{R}$$
$$\Rightarrow B(g,r)=\{f:\mathbb{R}\to\mathbb{R}; g(x)-r(x)<f(x)<g(x)+r(x)\ \ \forall x\in\mathbb{R}\}$$
Hence, the POR-open ball $B(g,r)$ consists of all functions lying within the open band bounded by the curves $g(x)+r(x)$ and $g(x)-r(x)$.
The POR-closed ball $\bar{B}(g,r)$ consists of all functions lying within the closed band bounded by the curves $g(x)+r(x)$ and $g(x)-r(x)$.
The exterior of the POR-open ball:
$$B^c(g,r)=\{f:\mathbb{R}\to\mathbb{R}; |f(x)-g(x)|\geq r(x)\ \ \forall x\in\mathbb{R}\}$$
$$=\{f:\mathbb{R}\to\mathbb{R}; f(x)\geq g(x)+r(x)\ or\ f(x)\leq g(x)-r(x)\ \ \forall x\in\mathbb{R}\}$$
The exterior of the POR-closed ball:
$$\bar{B}^c(g,r)=\{f:\mathbb{R}\to\mathbb{R}; f(x)>g(x)+r(x)\ or\ f(x)<g(x)-r(x)\ \ \forall x\in\mathbb{R}\}$$
The POR-sphere:
$$S(\,g,r)=\{f:\mathbb{R}\to\mathbb{R}; |f(x)-g(x)|=r(x)\ \forall x\in\mathbb{R}\}$$
$$=\{f:\mathbb{R}\to\mathbb{R}; f(x)=g(x)+r(x) or\ f(x)=g(x)-r(x)\ \forall x\in\mathbb{R}\}$$
So, the POR-sphere $S(\,g,r)$ is the set of the functions $g(x)+r(x)$, $g(x)-r(x)$ that forms the envelope of the band.
Theorem 4.1: In a POR-metric space it's follow that:
1. $B(a,r)\subseteq\bar{B}(a,r),\bar{B}^c(a,r)\subseteq B^c(a,r)$
2. $B(a,r)\cap B^c(a,r)=B(a,r)\cap\bar{B}^c(a,r)=\emptyset$

3. $\bar{B}(a,r) \cap B^c(a,r) = S(a,r)$
4. $B(a,r) \cup B^c(a,r) \neq X$, $\bar{B}(a,r) \cup \bar{B}^c(a,r) \neq X$ $(in\ general)$.

Proof:

1. $\forall x \in B(a,r)$ then, $d(x,a) < r$ which implies that $d(x,a) \leq r$.thus, $x \in \bar{B}(a,r)$.
   $\forall x \in \bar{B}^c(a,r)$ then,$d(x,a) > r$ which implies that $d(x,a) \geq r$.thus, $x \in B^c(a,r)$.
2. Suppose $x \in B(a,r) \cap B^c(a,r)$ then, $x \in B(a,r)$ and $x \in B^c(a,r)$, which gives, $d(x,a) < r$ and $d(x,a) \geq r$, which is a contradiction to the conditions of the partial order relation. So $B(a,r) \cap B^c(a,r) = \emptyset$.
   by the same argument we have that $B(a,r) \cap \bar{B}^c(a,r) = \emptyset$.
3. $\forall x \in \bar{B}(a,r) \cap B^c(a,r) \Leftrightarrow x \in \bar{B}(a,r)$ and $x \in B^c(a,r)$.
   $\Leftrightarrow d(x,a) \leq r$ and $d(x,a) \geq r$.
   By antisymmetry:
   $\Leftrightarrow d(x,y) = r \Leftrightarrow x \in S(a,r)$
4. Let's take the POR-metric space $(\mathbb{R}^2, d)$ over the PO-ring $\mathbb{R}(I)$.
   let's take $(1,-2) \in \mathbb{R}^2$, and $r = 2 + I \in \mathbb{R}(I)^+$.
   then, depending on what we proved in Example 4.1, we can see:
   $B\big((1,-2), 2+I\big) = \left\{(x_1,x_2) \in \mathbb{R}^2;\ \begin{cases} -1 < x_1 < 3 \\ -5 < x_2 < +1 \end{cases}\right\}$
   $B^c\big((1,-2), 2+I\big) = \left\{(x_1,x_2) \in \mathbb{R}^2;\ \begin{cases} |x_1 - 1| \geq 2 \\ |x_2 + 2| \geq 3 \end{cases}\right\}$
   now, take $x = (0,2) \in \mathbb{R}^2$ then $x \notin B\big((1,-2), 2+I\big)$
   and $x \notin B^c\big((1,-2), 2+I\big)$ so: $x \notin B\big((1,-2), 2+I\big) \cup B^c\big((1,-2), 2+I\big)$

Therefor, $B\big((1,-2), 2+I\big) \cup B^c\big((1,-2), 2+I\big) \neq \mathbb{R}^2$
Moreover, $x \notin \bar{B}\big((1,-2), 2+I\big) \cup \bar{B}^c\big((1,-2), 2+I\big)$
then, $\bar{B}\big((1,-2), 2+I\big) \cup \bar{B}^c\big((1,-2), 2+I\big) \neq \mathbb{R}^2$

Theorem 4.2: Let R be an L-PO-ring with a clear definition of $(<)$, and let (X,d) be a POR-metric space over R. Then, the following is satisfied:
for $a \in X$ and $r_1, r_2 \in R^+$:

1. $B(a,r_1) \cap B(a,r_2) \subseteq \bar{B}(a, inf(r_1,r_2))$. If $inf(r_1,r_2) \notin \{r_1,r_2\}$ then, $B(a,r_1) \cap B(a,r_2) = \bar{B}(a, inf(r_1,r_2))$.
2. $\bar{B}^c(a,r_1) \cap \bar{B}^c(a,r_2) \subseteq B^c(a, sup(r_1,r_2))$. If $sup(r_1,r_2) \notin \{r_1,r_2\}$ then, $B(a,r_1) \cap B(a,r_2) = \bar{B}(a, sup(r_1,r_2))$.
3. $B^c(a,r_1) \cap B^c(a,r_2) = B^c(a, sup(r_1,r_2))$.
4. $\bar{B}(a,r_1) \cap \bar{B}(a,r_2) = \bar{B}(a, inf(r_1,r_2))$.

Proof:

1. $\forall x \in B(a,r_1) \cap B(a,r_2)$ $then, x \in B(a,r_1)$ and $, x \in B(a,r_2)$
   so, $d(x,a) < r_1$and $d(x,a) < r_2$. Since R is an L-PO-ring then
   $d(x,a) \leq inf(r_1,r_2) \Rightarrow x \in \bar{B}(a, inf(r_1,r_2))$

Thus, $B(a,r_1) \cap B(a,r_2) \subseteq \bar{B}(a, inf(r_1,r_2))$
Now, for any $x \in \bar{B}(a, inf(r_1,r_2))$ then $d(x,a) \leq inf(r_1,r_2)$, but if $inf(r_1,r_2) \notin \{r_1,r_2\}$, then $inf(r_1,r_2) < r_1$ and $inf(r_1,r_2) < r_2$, so
$d(x,a) < r_1$and $d(x,a) < r_2$. Hence, $x \in B(a,r_1)$ and $x \in B(a,r_2)$

Thus, $x \in B(a, r_1) \cap B(a, r_2)$ then, $\bar{B}(a, inf(r_1, r_2)) \subseteq B(a, r_1) \cap B(a, r_2)$. With the first part we obtain that: $B(a, r_1) \cap B(a, r_2) = \bar{B}(a, inf(r_1, r_2))$.

2. It can be shown by the same way.
3. $\forall x \in B^c(a, r_1) \cap B^c(a, r_2)\ then, x \in B^c(a, r_1)\ and\ , x \in B^c(a, r_2)$
   so, $d(x, a) \geq r_1 and\ d(x, a) \geq r_2$. Since R is an L-PO-ring then
   $d(x, a) \geq \sup(r_1, r_2) \Rightarrow x \in B^c(a, sup(r_1, r_2))$.
   Thus, $B^c(a, r_1) \cap B^c(a, r_2) \subseteq B^c(a, sup(r_1, r_2))$.
   On the other hand, let $x \in B^c(a, sup(r_1, r_2))$ then $d(x, a) \geq sup(r_1, r_2)$ therefor, $d(x, a) \geq r_1$ and $d(x, a) \geq r_2$ which leads to: $x \in B^c(a, r_1)$ and $x \in B^c(a, r_2)$. Then $B^c(a, sup(r_1, r_2)) \subseteq B^c(a, r_1) \cap B^c(a, r_2)$.
   By the two inclusion, the required is done.
4. It can be shown by the same way.

Corollary 4.1: For any $x \in X$ we have that $\{x\}$ is a POR-closed ball.
Proof: take the POR-closed ball with the center x and the radius 0, then:
$\bar{B}(x, 0) = \{y \in X\ ; d(x, y) \leq 0\}$, but $d(x, y) \in R^+$ so :
$\bar{B}(x, 0) = \{y \in X\ ; d(x, y) = 0\} = \{x\}$

Definition 4.1: Let (X, d) be a POR-metric space over the ring R, let $(S, +)$ be a subgroup of $(R, +)$ we define the relation $\sim_s$ as follows:

$$For\ any\ x, y \in X\ : x \sim_s y \Leftrightarrow \exists a \in S\ ; d(x, y) \leq a$$

Theorem 4.3: the relation $\sim_s$ is an equivalence relation.
Proof:
Reflexivity: Since $d(x, x) = 0$ and $0 \in S$ then there is an element $0 \in S$ such that $d(x, x) \leq 0$ thus, $x \sim_s x$
symmetry**:** Suppose that $x \sim_s y$ then $\exists a \in S\ ; d(x, y) \leq a$ and by the first condition of the POR-metric we have $d(y, x) = d(x, y) \leq a$, then, $y \sim_s x$
Transitivity: **:** Suppose that $x \sim_s y$ and $y \sim_s z$ so there exist two elements $a, b \in S$ such that: $d(x, y) \leq a\ and\ \ d(y, z) \leq b$. And by the second condition of the POR-metric we have:
$d(x, z) \leq d(x, y) + d(y, z) \leq a + b$.
And as S is a subgroup then $a + b \in S$ hence, $d(x, z) \leq a + b$ and so $x \sim_s z$.
Therefore, $\sim_s$ is an equivalence relation.
Remark 4.2: We denote the equivalence classes of $\sim_s$ by:

$$[x]_S = \{y \in X\ ; x \sim_s y\}$$

We call $\{[x]_S\ ; x \in X\}$ the S-partition of (X, d).
Theorem 4.4: For any POR-metric space (X, d) over R , for all $x \in X$, the following is satisfied, where $S\ and\ K$ are two subgroups of $(R, +)$:

1. $[x]_{S \cap K} \subseteq [x]_S \cap [x]_K$, $[x]_S \cup [x]_K \subseteq [x]_{S+K}$ where:
   $S + K = \{a + b\ ; a \in S\ \ , b \in K\}$
2. If $r \in S$ then $B(a, r) \subseteq \bar{B}(a, r) \subseteq [a]_S$

Proof:

1. $\forall y \in [x]_{S \cap K}\ \Rightarrow x \sim_{S \cap K} y\ \Leftrightarrow\ \exists a \in S \cap K\ ; d(x, y) \leq a$
   this implies that: $a \in S$ with $d(x, y) \leq a$ then $x \sim_s y$
   and $a \in K$ with $d(x, y) \leq a$ then, $x \sim_K y$.
   Thus, $y \in [x]_S \cap [x]_K$

$\forall y \in [x]_S \cup [x]_K \Rightarrow y \in [x]_S \; or \; y \in [x]_K$ so,
$x \sim_s y \; or \; x \sim_K y \Rightarrow \exists a \in S \; ; \; d(x, y) \leq a \; or \; b \in K \; ; \; d(x, y) \leq b.$
since $S \subseteq S + K \; and \; K \subseteq S + K$ then, $a, b \in S + K$
so wither $d(x, y) \leq a \; or \; d(x, y) \leq b \Rightarrow x \sim_{S+K} y$
Thus, $y \in [x]_{S+K}$.

2. $\forall x \in \bar{B}(a, r) \Rightarrow d(x, a) \leq r$ and since $r \in S$ then $a \sim_s x$ so $x \in [a]_S$

And $B(a, r) \subseteq \bar{B}(a, r)$ has been proved in the previous pages.

5. Basic topological concepts:

Definition 5.1: Let (X, d) be a POR-metric space over the PO-ring R, for any $x \in X$ we called the POR-open ball $B(x, r)$, where $r \in R$, a POR-open neighborhood of x, and the set of all POR-open neighborhoods of x is denoted by POR-$V(x)$.

Definition 5.2: Let A be a subset of the POR-metric space (X, d) over the PO-ring R, a point $x \in A$ is called a POR-interior point of A if and only if there exists a POR-open neighborhood of x which is contained in A.
i.e. : $\exists v \in$ POR-$V(x)$ such that $x \in v \subseteq A$.
Furthermore, the set of all such points is called the POR-interior of A, denoted by $Int_{POR}(A)$.

Corollary 5.1: For any set $A \subseteq X$ it is obvious from the definition that:

$$Int_{POR}(A) \subseteq A \; .$$

Definition 5.3: Let A be a subset of (X, d), let x be an element of X, we call x a POR-closure point of A if and only if for any POR-open neighborhood of x, v, there is an element belongs to the intersection of A and v.
i.e.: $\forall v \in$ POR-$V(x)$ then $v \cap A \neq \emptyset$.
We denote the set of all POR-closure points of A by $Cl_{POR}(A)$.

Corollary 5.2: For any set $A \subseteq X$ then,

$$A \subseteq Cl_{POR}(A) \text{ and } Int_{POR}(A) \subseteq Cl_{POR}(A)$$

Proof: let x be an element of A**,** then for any $v \in$ POR-$V(x)$, $x \in v$, so $x \in v \cap A$. Hence, $v \cap A \neq \emptyset$, and $x \in Cl_{POR}(A)$.
the second conclusion is a direct result from Corollary 5.1and the first conclusion.

Definition 5.4: Let A be a subset of (X, d), let x be an element of X, we call x a POR-limit point of A if and only if for any POR-open neighborhood of x, v, the intersection of A and v contains at least an element different from x.
i.e.: $\forall v \in$ POR-$V(x)$ then $v \cap A \neq \emptyset$ and $v \cap A \neq \{x\}$.
We denote the set of all POR-limit points of A by $\acute{A}_{POR}$.

Corollary 5.3: For any set $A \subseteq X$ then, $\acute{A}_{POR} \subseteq Cl_{POR}(A)$**.**
Proof: It is obvious from the definition.

Proposition 5.1: For any set $A \subseteq X$ then, $Cl_{POR}(A) = A \cup \acute{A}_{POR}$.

Proof: from the previous corollaries, we have that $\acute{A}_{POR} \subseteq Cl_{POR}(A)$ and $A \subseteq Cl_{POR}(A)$ then, $A \cup \acute{A}_{POR} \subseteq Cl_{POR}(A)$.
From the other side, take $x \in Cl_{POR}(A)$ then, if $x \in A$ we deduce that $x \in A \cup \acute{A}_{POR}$.
If $x \notin A$, then $x \notin v \cap A$ for any $v \in$ POR-$V(x)$ so, $v \cap A \neq \{x\}$ $\forall v \in$POR-$V(x)$, moreover, since $x \in Cl_{POR}(A)$ then, $\forall v \in$ POR-$V(x)$ then $v \cap A \neq \emptyset$. Hence, $x \in \acute{A}_{POR}$ and so, $x \in A \cup \acute{A}_{POR}$. This implies that $Cl_{POR}(A) \subseteq A \cup \acute{A}_{POR}$. From the tow conclusions, we get the required.

Definition 5.5: Let A be a subset of (X, d), let x be an element of A, we call x a POR-isolated point of A if and only if there exists a POR-open neighborhood of x such that x is the unique element of the intersection of A and v.
i.e. : $\exists v \in$ POR-$V(x)$ such that $v \cap A = \{x\}$.
We denote the set of all POR-isolated points of A by $Is_{POR}(A)$.

Corollary 5.4: For any set $A \subseteq X$ then, $Is_{POR}(A) \subseteq Cl_{POR}(A)$.
Proof: it is clear from the definitions.

Proposition 5.2: For any set $A \subseteq X$ then, $\acute{A}_{POR} = Cl_{POR}(A) \backslash Is_{POR}(A)$.
Proof: since $Is_{POR}(A) \subseteq Cl_{POR}(A)$ and $\acute{A}_{POR} \subseteq Cl_{POR}(A)$, and by the definitions 5.5 and 5.4 we can notice that $\acute{A}_{POR} \cap Is_{POR}(A) = \emptyset$.
Hence, $\acute{A}_{POR} \subseteq Cl_{POR}(A) \backslash Is_{POR}(A)$.
Take $x \in Cl_{POR}(A) \backslash Is_{POR}(A)$ then $x \in Cl_{POR}(A)$ and $x \notin Is_{POR}(A)$.
Which implies that $v \cap A \neq \emptyset$ for every $v \in$ POR-$V(x)$ and $v \cap A \neq \{x\}$ for every $v \in$ POR-$V(x)$, we deduce that:
$\forall v \in$ POR-$V(x)$ then $v \cap A \neq \emptyset$ and $v \cap A \neq \{x\}$.
Thus, $x \in \acute{A}_{POR}$ and $Cl_{POR}(A) \backslash Is_{POR}(A) \subseteq \acute{A}_{POR}$. from the tow conclusions we get the required.

Definition 5.6: Let A be a subset of (X, d), let x be an element of X, we call x a POR-boundary point of A if and only if x is a POR- closure point of A and a POR- closure point of $X \backslash A$.
i.e. : $\forall v \in$ POR-$V(x)$ then $v \cap A \neq \emptyset$ and $v \cap (X \backslash A) \neq \emptyset$
We denote the set of all POR- boundary points of A by $bd_{POR}(A)$.

Corollary 5.5: For any set $A \subseteq X$ then, $bd_{POR}(A) = Cl_{POR}(A) \cap Cl_{POR}(X \backslash A)$.
Proof: it is clear from the definitions.

6. Open and closed sets in POR-metric space

Definition 6.1: Let A be a subset of (X, d), it's said to be a POR-open set if and only if $Int_{POR}(A) = A$.

Theorem 6.1: Every POR-open ball is a POR-open set.
Proof:
Let (X, d) be a POR-metric space, for any POR-open ball $B(a, r)$ we have that $Int_{POR}\big(B(a, r)\big) \subseteq B(a, r)$, now take an arbitrary element $x \in B(a, r)$ then $d(x, a) < r \Rightarrow r - d(x, a) > 0$, take $\rho = r - d(x, a) > 0$ then there exists the POR-open

ball $B(x,\rho)$ where $x \in B(x,\rho) \subseteq B(a,r)$ to prove that, take any $y \in B(x,\rho)$ then,
$d(y,x) < \rho$
for the elements x, y, a we have that:
$d(y,a) \leq d(y,x) + d(x,a) < \rho + d(x,a) = r - d(x,a) + d(x,a)$ thus:
$d(y,a) < r$ which implies that $y \in B(a,r)$ and so, $B(x,\rho) \subseteq B(a,r)$
Therefore, x is a POR-interior point of $B(a,r)$, which gives that:
$B(a,r) \subseteq Int_{POR}\big(B(a,r)\big)$. In the result $B(a,r)$ is a POR-open set.

We give the following example to clarify that the converse of the Theorem 6.1 is not necessary hold

Example 6.1: take the space $(M_2(\mathbb{R}), d)$ of all 2×2 matrices over $\mathbb{R}$, under the addition, multiplication and the partial order relation with the POR-metric defined in Example 3.4
Let's take the subset A of $M_2(\mathbb{R})$ such that:
$M = \left\{\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix};\ |a_1| < 1\,, |a_4| < 1\ where\ a_1, a_2, a_3, a_4 \in \mathbb{R}\right\}$
We prove that M is a POR-open set but not a POR-open ball:
for any $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in M\ then\ |a_1| < 1\,, |a_4| < 1$ thus,
$1 - |a_1| > 0\ and\ 1 - |a_4| > 0$, then there exist $R = \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix} \in \big(M_2(\mathbb{R})\big)^+$
where $r_1 = \frac{1-|a_1|}{2}$, $r_4 = \frac{1-|a_4|}{2}$ and $r_2, r_3 \in \mathbb{R}^+$such that :
$\forall X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in B(A,R)$ we deduced that $X \in M$, this is because:
$|x_i - a_i| < r_i$ for every $i \in \{1,2,3,4\}$ thus for $i \in \{1,4\}$: $|x_i - a_i| < \frac{1-|a_i|}{2}$
Which implies that: $|x_i| - |a_i| < \frac{1-|a_i|}{2}$ then : $|x_i| < \frac{1+|a_i|}{2} < \frac{2}{2}$
hence, $|x_i| < 1$ for $i \in \{1,4\}$
We obtained that for any $A \in M$ there exists a POR-open ball $B(A,R)$ such that :
$A \in B(A,R) \subseteq M$ then $M \subseteq Int_{POR}(M)$ and $Int_{POR}(M) \subseteq M$ then M is equal to its interior so it's a POR-open set.
However, M can't be expressed as a POR-open ball, since its definition does not match the structure of POR-open balls which require a uniform radius R around a fixed center.

Theorem 6.2: Let A be a subset of a POR-metric space, then A is a POR-open set if and only if A is a union of POR-open balls.
Proof:
let A be a POR-open set then $Int_{POR}(A) = A$ then for any $a \in A$ there exist a POR-open ball $B(a, r_a)$ such that $\{a\} \subseteq B(a, r_a) \subseteq A$ thus,
$\bigcup_{a\in A}\{a\} \subseteq \bigcup_{a\in A} B(a, r_a) \subseteq A$ hence, $A = \bigcup_{a\in A} B(a, r_a)$.

Conversely, let the subset A of X with $A = \bigcup_{i\in I} B(x_i, r_i)$ then, for any $a \in A$ there exist an $i_0 \in I$ where $a \in B\big(x_{i_0}, r_{i_0}\big)$.
By the Theorem 6.1 we have $Int_{POR}\left(B\big(x_{i_0}, r_{i_0}\big)\right) = B\big(x_{i_0}, r_{i_0}\big)$ then,
$a \in Int_{POR}\left(B\big(x_{i_0}, r_{i_0}\big)\right)$ which implies that there exist a POR-open ball $B(a,\rho)$such that $a \in B(a,\rho) \subseteq B\big(x_{i_0}, r_{i_0}\big)$ and since $B\big(x_{i_0}, r_{i_0}\big) \subseteq A$ then,

$a \in B(a,\rho) \subseteq A$. Hence, a is a POR-interior point of A, and $A \subseteq Int_{POR}(A)$ and the reverse inclusion always holds, then, A is a POR-open set.

Proposition 6.1: The union of any POR-open sets is a POR-open set.
Proof: Suppose A_i for any $i \in I$ are POR-open sets. Then $Int_{POR}(A_i) = A_i$ for any $i \in I$, we need to prove that $Int_{POR}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} A_i$.
we have that $Int_{POR}(\bigcup_{i\in I} A_i) \subseteq \bigcup_{i\in I} A_i$, let's prove the other inclusion. Take any $x \in \bigcup_{i\in I} A_i$ then, there exist $i_0 \in I$ such that $x \in A_{i_0}$ hence, $x \in Int_{POR}(A_{i_0})$. By the definition of the interior of A_{i_0}, there exist a radius r satisfying that: $x \in B(x,r) \subseteq A_{i_0}$, and since $A_{i_0} \subseteq \bigcup_{i\in I} A_i$, it follows that $x \in B(x,r) \subseteq \bigcup_{i\in I} A_i$. Thereby we prove that $x \in Int_{POR}(\bigcup_{i\in I} A_i)$, then by two inclusion we deduce that $Int_{POR}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} A_i$.

Theorem 6.3: Let A be a subset of the POR-metric space (X,d) over the PO-ring R, and let Ω be the family of all POR-open sets of X contained in A
i.e. $\Omega = \{U_i \,; i \in I, U_i \text{ is } POR - open\ set\ , U_i \subseteq A\}$, then $Int_{POR}(A) = \bigcup_{i\in I} U_i$.
Proof: take $x \in Int_{POR}(A)$ then by the definition there exist $v \in$ POR-$V(x)$ such that $x \in v \subseteq A$ and due to the definition of the POR-neighborhoods $v = B(x,r)$ which is a POR-open set , thus $v \in \Omega$ so $v \in \bigcup_{i\in I} U_i$.
For the next step, let's take $x \in \bigcup_{i\in I} U_i$, by
Proposition **6.1** $\bigcup_{i\in I} U_i$ is a POR-open set, and since $U_i \subseteq A$, then $\bigcup_{i\in I} U_i \subseteq A$. Hence $x \in \bigcup_{i\in I} U_i \subseteq A$ where $\bigcup_{i\in I} U_i$ is POR-open, this yields that $x \in Int_{POR}(A)$ and then $\bigcup_{i\in I} U_i \subseteq Int_{POR}(A)$. As a result of the above, we obtain that $Int_{POR}(A) = \bigcup_{i\in I} U_i$.

Definition 6.5: Let A be a subset of X where (X,d) is a POR-metric space. We call A a POR-closed set if and only if : $\acute{A}_{POR} \subseteq A$.

Theorem 6.4: Let $A \subseteq X$, where (X,d) is a POR-metric space. Then A is a POR-closed set if and only if $X\backslash A$ is a POR-open set.
Proof: If A is a POR-closed set then, $\acute{A}_{POR} \subseteq A$, take an arbitrary element $x \in X\backslash A$ we need to prove that there exist a radius $r \in R^+$ such that $x \in B(x,r) \subseteq X\backslash A$, assume that there isn't any radius r satisfying that $B(x,r) \subseteq X\backslash A$, therefore, $B(x,r) \nsubseteq X\backslash A$ for any $r \in R^+$. This implies that for any radius r there is an element y belongs to $B(x,r)$ and y doesn't belong to $X\backslash A$, in other words, $\forall r \in R^+\ \exists y \in B(x,r)$ and $y \in A$ (it is obvious that $y \neq x$).
It follows that $y \in B(x,r) \cap A$ for all $r \in R^+$. Hence, $B(x,r) \cap A \neq \emptyset$ and $B(x,r) \cap A \neq \{x\}$ for all $r \in R^+$, this yields that $x \in \acute{A}_{POR}$, but $\acute{A}_{POR} \subseteq A$ then $x \in A$, this contradicts the fact that $x \in X\backslash A$. We conclude that there exist $r \in R^+$ such that $x \in B(x,r) \subseteq X\backslash A$ for all $x \in X\backslash A$ and so $X\backslash A$ is a POR-open set.
To prove the convers, let $X\backslash A$ be a POR-open set, for any $x \in \acute{A}_{POR}$ assume that $x \notin A$ then $x \in X\backslash A$ which is a POR-open set, therefore, there exist $r_0 \in R^+$ such that $x \in B(x,r_0) \subseteq X\backslash A$, this implies that for any $y \in B(x,r_0)$, with $y \neq x$, then, $y \in X\backslash A$. But $B(x,r_0)$ is a POR-open neighborhood of x, and for $y \in B(x,r_0)$, we get that $y \notin A \cap B(x,r_0)$. This is a contradiction with the assumption that $x \in \acute{A}_{POR}$.
Consequently, $x \in A$ and $\acute{A}_{POR} \subseteq A$.

Remark 6.1: It is not a necessary for the POR-closed ball to be a POR-closed set, and the next example shows that.

Example 6.2: Let's consider the POR-metric space $(\mathbb{R}\times\mathbb{R}, d)$ where d is the POR-metric defined over the PO-ring $\mathbb{Z}(I)=\{a+bI\,;I^2=I\,;a,b\in\mathbb{Z}\}$ as follows:
$d\big((x,y),(z,t)\big)=|x-z|+|y-t|I$
Let's define the POR-closed ball $\bar{B}\big((0,0),1+I\big)=\big\{(x_1,x_2);d\big((x_1,x_2),(0,0)\big)\le 1+I\,\big\}=\{(x_1,x_2);|x_1|\le 1,|x_2|\le 1\,\}$
The next step is to choose a point which doesn't belong to the POR-closed ball but it is a POR-limit point of that ball.
Choose the point $\left(0,\frac{3}{2}\right)\notin\bar{B}\big((0,0),1+I\big)$, for any $r_1+r_2I\in\mathbb{Z}(I)^+$
$(0,2)\in B\left(\left(0,\frac{3}{2}\right),\,r_1+r_2I\right)=\left\{(y_1,y_2);|y_1|<r_1,\left|y_2-\frac{3}{2}\right|<r_2\,\right\}$ then, whatever r_1 and r_2 are, we can find the point $(0,1)$ satisfying that $(0,1)\in B\left(\left(0,\frac{3}{2}\right),\,r_1+r_2I\right)$ and $(0,1)\in\bar{B}\big((0,0),1+I\big)$. Hence, the intersection of $B\left(\left(0,\frac{3}{2}\right),\,r_1+r_2I\right)$ and $\bar{B}\big((0,0),1+I\big)$ contains the point $(0,1)$ for any $r_1+r_2I\in\mathbb{Z}(I)^+$, therefore, the point $\left(0,\frac{3}{2}\right)$ is a POR-limit point of the ball $\bar{B}\big((0,0),1+I\big)$, however, $\left(0,\frac{3}{2}\right)\notin\bar{B}\big((0,0),1+I\big)$ which implies that $\bar{B}\big((0,0),1+I\big)$ is not a POR-closed set.

Proposition 6.2: Assume that A_i for any $i\in I$ are POR-closed sets, then $\bigcap_{i\in I}A_i$ is a POR-closed set.
Proof: since A_i are POR-closed sets for any $i\in I$ then by Theorem 6.4 $X\backslash A_i$ are POR-open sets, consider $X\backslash\bigcap_{i\in I}A_i=\bigcup_{i\in I}X\backslash A_i$, and by the Proposition **6.1** the set $\bigcup_{i\in I}X\backslash A_i$ is a POR-open set. Consequently, $X\backslash\bigcap_{i\in I}A_i$ is a POR-open set, and then $\bigcap_{i\in I}A_i$ is a POR-closed set.

Corollary 6.1: Let $A\subseteq X$, where (X,d) is a POR-metric space. Then A is a POR-closed set if and only if $A=Cl_{POR}(A)$.
Proof: it can be easily shown using the Proposition 5.1.

Theorem 6.5: Let A be a subset of the POR-metric space (X,d) over the PO-ring R, and let Ω be the family of all POR-closed sets of X containing A
i.e. $\Omega=\{F_i\,;i\in I,F_i\ is\ POR-closed\ set\ ,A\subseteq F_i\}$, then $Cl_{POR}(A)=\bigcap_{i\in I}F_i$.
Proof: take $x\in X\backslash Cl_{POR}(A)$ i.e. $x\notin Cl_{POR}(A)$ which implies that there is a POR-open neighborhood of x as v such that $v\cap A=\emptyset$, thus $A\subseteq X\backslash v=F_0$ which is a POR-closed set because its complement is a POR-open set.
Hence, $F_0\in\Omega$ but $x\notin F_0$ as $x\in v$, we can now see that $x\notin\bigcap_{i\in I}F_i$.
Thereby $X\backslash Cl_{POR}(A)\subseteq X\backslash\bigcap_{i\in I}F_i$ and so $\bigcap_{i\in I}F_i\subseteq Cl_{POR}(A)$.
Next, take $x\in X\backslash\bigcap_{i\in I}F_i$ then $x\notin\bigcap_{i\in I}F_i$, it follows that there exist a set $F_1\in\Omega$ such that $x\notin F_1$, then $x\in X\backslash F_1=v_1$ which is a POR-open set, thus, $v_1\in V(x)$.
However, $F_1\in\Omega$, then $A\subseteq F_1$, therefore, $A\cap X\backslash F_1=A\cap v_1=\emptyset$, this leads to the result : $x\notin Cl_{POR}(A)$, now we obtain that $X\backslash\bigcap_{i\in I}F_i\subseteq X\backslash Cl_{POR}(A)$, and so $Cl_{POR}(A)\subseteq\bigcap_{i\in I}F_i$.
By the previous results we conclude that $Cl_{POR}(A)=\bigcap_{i\in I}F_i$.

7. The POR- supra-topology

Lemma 7.1: Let (X, d) be a L-POR-metric space over the PO-ring R, then the collection defined as follows:
$\tau^* = \{T \in \mathcal{P}(X)\ ;\ T\ is\ a\ union\ of\ POR - open\ balls\ of\ (X, d)\ \}$ (7.1) forms a supra-topology**.**

Proof: Take $T_i \in \tau^*\ \ \forall i \in I$ then, T_i is a union of POR-open balls of (X, d), we conclude that $\bigcup_{i\in I} T_i$ is also a union of POR-open balls of (X, d), hence, $\bigcup_{i\in I} T_i \in \tau^*$
As a result of the above, the collection τ^* is a supra-topology

Definition 7.1: Let (X, d) be a POR-metric space over the PO-ring R, then the collection τ^* defined in (7.1) is the supra-topology generated by the POR-metric d, we call it the POR-supra-topology generated by d, and denoted by τ^*_{POR-d}.

Theorem 7.1: Every POR-metric space is a T_1 space.

Proof: let (X, d) be a POR-metric space, for any distinct points $x, y \in X$ then, $d(x, y) > 0$, take $r = d(x, y)$ then $x \in B(x, r)$ but $y \notin B(x, r)$ because $d(x, y) \nless r$,
and $y \in B(y, r)$ but $x \notin B(y, r)$ because $d(x, y) \nless r$. It follows that (X, d) is a T_1 space, consequently, it is a T_0 space.

Theorem 7.2: Assume that R is a PO-ring with the property φ, and an element a which is a unit in the R satisfying that $a > 1$ such that $0 < 2a^{-1} < 1$. Let (X, d) be a POR-metric space over the previous ring R then, (X, d) is T_2 space.

Proof: For any $x \neq y \in X$ then $d(x, y) > 0$, putting $\rho = a^{-1}.d(x, y)$ then $\rho < d(x, y)$ that is because $a > 1$ so $a^{-1} < 1$. Note that the POR-open balls $B(x, \rho)\ and\ B(y, \rho)$ are disjoint, because, if we take any $z \in B(x, \rho) \cap B(y, \rho)$ then $d(x, z) < \rho$ and $d(y, z) < \rho$. But due to the triangle inequality $d(x, y) \leq d(x, z) + d(z, y) < \rho + \rho = 2\rho = 2a^{-1}.d(x, y)$. Thus, $d(x, y) < 2a^{-1}d(x, y)$, but $0 < 2a^{-1} < 1$ then, $2a^{-1}d(x, y) < d(x, y)$, so it is a contradiction , therefor, $B(x, \rho) \cap B(y, \rho) = \emptyset$.

Conclusions and Recommendations:

This work developed the concept of POR-metric spaces and examined the main topological features generated by their associated balls. The obtained results show that this framework offers a consistent extension of classical metric spaces while allowing the interaction between order and algebraic structures to play a central role.

For future research, several directions may be considered. In particular, it would be of interest to investigate continuity of mappings between POR-metric spaces and to study compactness and related notions within this framework. Further work may also explore fixed point results, as well as deeper connections between the algebraic properties of the underlying ring and the induced topological behavior.

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