

Fuzzy Strong Ideals In Bci-Algebras

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□ ABSTRACT □

In this paper, we introduce and study the notion of fuzzy strong ideals in BCI -algebras. This notion is motivated by the classical concept of strong ideals and by the level subset method used in the theory of fuzzy ideals. The aim of the paper is to provide a natural fuzzy extension of strong ideals, in which the classical strong property is reflected through the upper level subsets associated with a fuzzy subset.

We prove that a fuzzy subset is a fuzzy strong ideal if and only if all of its non-empty upper level subsets are strong ideals of the underlying algebra. We also show that the subset of elements having the same membership value as the zero element forms a strong ideal. In addition, we prove that this notion is essentially non-trivial in BCI -algebras, since every fuzzy strong ideal in a BCK -algebra is constant.

Furthermore, we investigate the relationship between strong ideals and characteristic functions, and we provide a characterization of two-valued fuzzy strong ideals. Homomorphic preimages of fuzzy strong ideals are also studied, and it is shown that the family of fuzzy strong ideals has a meet-semilattice structure with respect to the pointwise meet operation. Several examples are included to illustrate the main results and to distinguish fuzzy strong ideals from ordinary fuzzy ideals.

Keywords: BCI -algebra, BCK -algebra, ideal, strong ideal, fuzzy ideal, fuzzy strong ideal, level subset, $\wedge$ -lattice.

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المثاليات الضبابية القوية في جبر BCI

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□ ملخص □

في هذا البحث، نعرّف وندرس مفهوم المثاليات الضبابية القوية في جبر BCI ، وذلك اعتماداً على المفهوم الكلاسيكي للمثاليات القوية وعلى طريقة مجموعات المستوى المستخدمة في نظرية المثاليات الضبابية. يهدف البحث إلى تقديم توسيع ضبابي طبيعي للمثالية القوية، بحيث تتعكس خاصية القوة الكلاسيكية على جميع مجموعات المستوى العليا المرتبطة بالمجموعة الضبابية.

نثبت في هذا العمل أن المجموعة الضبابية تكون مثالية ضبابية قوية إذا وفقط إذا كانت جميع مجموعات المستوى العليا غير الخالية لها مثاليات قوية في الجبر المدروس. كما نبين أن مجموعة العناصر التي تأخذ القيمة العظمى نفسها لقيمة الصفر تشكل مثالية قوية. بالإضافة إلى ذلك، نوضح أن هذا المفهوم يكون غير تافه بصورة أساسية في جبر BCI ، إذ إن كل مثالية ضبابية قوية في جبر BCK تكون ثابتة.

كما ندرس العلاقة بين المثاليات القوية والدوال المميزة، ونقدم توصيفاً للمثاليات الضبابية القوية ثنائية القيم. ونبحث أيضاً في سلوك المثاليات الضبابية القوية تحت الصور العكسية للهومومورفيزمات، ونثبت أن عائلة المثاليات الضبابية القوية تمتلك بنية نصف شبكة بالنسبة لعملية الالتقاء النقطية. وتُرفق الدراسة بأمثلة توضيحية تبين الفروق بين المثاليات الضبابية والمثاليات الضبابية القوية.

الكلمات المفتاحية: جبر BCI ، جبر BCK ، مثالية، مثالية قوية، مثالية ضبابية، مثالية ضبابية قوية ، مجموعة مستوى، ٨- شبكة.



حقوق النشر : مجلة جامعة اللاذقية (تشرين سابقاً) - سورية، يحتفظ المؤلفون بحقوق النشر

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1. Introduction:

BCK-algebras were introduced by Imai and Iséki, and the related BCI framework was subsequently developed by Iséki [1,2]. Several classes of ideals have since been studied in BCK/BCI-algebras; in particular, strong ideals and their basic properties were investigated in [7,8].

Following Zadeh's introduction of fuzzy sets [3] and Rosenfeld's fuzzy algebraic approach [4], ideal theory in BCK/BCI-algebras was extended to the fuzzy setting [5,6]. The level-subset method is central to these extensions because it connects membership inequalities with the corresponding classical ideal properties [5,6].

Recent studies have developed interval-valued m -polar fuzzy ideals [9], tripolar picture fuzzy ideals [10], bipolar intuitionistic fuzzy soft ideals [11], and complex fuzzy ideals [12] in BCK/BCI-algebras. These works demonstrate the continuing development of fuzzy ideal theory. The present paper addresses a complementary question: the direct level-subset extension of the classical strong-ideal condition.

Let I be a strong ideal of a BCI-algebra X . Its defining condition can be written as

$$\text{If } x \in I \text{ and } x * a \in I, \text{ then } a \in I, \text{ for all } x, a \in X.$$

For a fuzzy subset $f: X \rightarrow [0,1]$, put $U(f,t) = \{x \in X : f(x) \geq t\}$. Requiring every non-empty upper level subset $U(f,t)$ to be a strong ideal yields

$$\text{If } f(x) \geq t \text{ and } f(x * a) \geq t, \text{ then } f(a) \geq t.$$

Equivalently,

$$f(a) \geq \min\{f(x), f(x * a)\}, \text{ for all } x, a \in X.$$

This inequality motivates the fuzzy strong condition introduced below.

The principal results establish the upper level characterization, prove that $U(f, f(0))$ is a strong ideal, show that fuzzy strong ideals on BCK-algebras are necessarily constant, characterize strong ideals by two-valued characteristic fuzzy subsets, prove stability under homomorphic preimages, and establish closure under pointwise meet.

2. Preliminaries

Definition 2.1. [1,2] Let X be a non-empty set with a binary operation $*$ and a distinguished element 0 . The structure $(X, *, 0)$ is called a BCI-algebra if the following conditions hold for all $x, y, z \in X$:

$$(I) \quad ((x * y) * (x * z)) * (z * y) = 0,$$

$$(II) \quad (x * (x * y)) * y = 0,$$

$$(III) \quad x * x = 0,$$

$$(IV) \quad \text{If } x * y = 0 \text{ and } y * x = 0, \text{ then } x = y.$$

A BCI-algebra X is called a BCK-algebra if it also satisfies

$$(V) \quad 0 * x = 0, \quad \forall x \in X.$$

Definition 2.2. [2] Let X be a BCI-algebra. We define a relation $\leq$ on X by

$$x \leq y \Leftrightarrow x * y = 0.$$

Definition 2.3. [5,6] A non-empty subset I of a BCI-algebra X is called an ideal of X if:

$$(I_1) \quad 0 \in I,$$

$$(I_2) \quad \text{If } x * y \in I \text{ and } y \in I, \text{ then } x \in I, \text{ for all } x, y \in X.$$

Definition 2.4. [7,8] Let X be a BCI-algebra and let I be an ideal of X . Then I is called a strong ideal of X if

$$(S) \quad \text{If } x \in I \text{ and } a \notin I, \text{ then } x * a \notin I, \text{ for all } x, a \in X.$$

Equivalently,

$$\text{If } x \in I \text{ and } x * a \in I, \text{ then } a \in I, \text{ for all } x, a \in X.$$

Definition 2.5. [3] Let X be a non-empty set. A fuzzy subset of X is a function

$$f: X \rightarrow [0,1].$$

Definition 2.6. [5,6] Let X be a BCI-algebra. A fuzzy subset $f: X \rightarrow [0,1]$ is called a fuzzy ideal of X if:

$$(F_1) \quad f(0) \geq f(x), \quad \forall x \in X,$$

$$(F_2) \quad f(x) \geq \min\{f(x * y), f(y)\}, \quad \forall x, y \in X.$$

Definition 2.7. [5,6] Let $f: X \rightarrow [0,1]$ be a fuzzy subset. For each $t \in [0,1]$, define

$$U(f, t) = \{x \in X : f(x) \geq t\}.$$

3. Fuzzy Strong Ideals

Motivation 3.1. Definition 3.2 is constructed by combining the classical strong-ideal condition in Definition 2.4 [7,8] with the upper level-subset method used for fuzzy ideals [5,6]. The classical condition is

$$\text{If } x \in I \text{ and } x * a \in I, \text{ then } a \in I, \text{ for all } x, a \in X.$$

For a fuzzy subset f and a threshold t , the statements $x \in U(f, t)$ and $x * a \in U(f, t)$ mean $f(x) \geq t$ and $f(x * a) \geq t$, respectively. Therefore, requiring every non-empty $U(f, t)$ to satisfy the classical strong-ideal condition gives

$$\text{If } f(x) \geq t \text{ and } f(x * a) \geq t, \text{ then } f(a) \geq t.$$

If this implication holds for every t , take $t = \min\{f(x), f(x * a)\}$. Conversely, the inequality below immediately implies the preceding threshold condition for every t . Thus the two formulations are equivalent:

$$f(a) \geq \min\{f(x), f(x * a)\}, \text{ for all } x, a \in X.$$

Consequently, (FS3) is obtained systematically from the classical strong-ideal property through upper level subsets; together with the usual fuzzy-ideal conditions (FS1) and (FS2), this yields Definition 3.2.

Definition 3.2. Let X be a BCI-algebra. A fuzzy subset $f: X \rightarrow [0,1]$ is called a fuzzy strong ideal of X if:

$$(FS_1) \quad f(0) \geq f(x), \quad \forall x \in X,$$

$$(FS_2) \quad f(x) \geq \min\{f(x * y), f(y)\}, \quad \forall x, y \in X,$$

$$(FS_3) \quad f(a) \geq \min\{f(x), f(x * a)\}, \quad \forall x, a \in X.$$

The condition (FS_3) is called the fuzzy strong condition.

Remark 3.3. Conditions (FS1) and (FS2) say exactly that f is a fuzzy ideal, while (FS3) is the additional strongness condition derived above from the classical strong-ideal property.

It follows directly from Definition 3.2 that every fuzzy strong ideal of a BCI-algebra X is a fuzzy ideal of X .

Theorem 3.4. Let X be a BCI-algebra and let f be a fuzzy strong ideal of X . Then

$$f(x) = f(0) \text{ and } f(x * a) = f(0) \Rightarrow f(a) = f(0), \quad \forall x, a \in X.$$

Proof. Since f is fuzzy strong,

$$f(a) \geq \min\{f(x), f(x * a)\}.$$

If $f(x) = f(0)$ and $f(x * a) = f(0)$, then

$$f(a) \geq \min\{f(0), f(0)\} = f(0).$$

By (FS_1) , $f(0) \geq f(a)$. Hence $f(a) = f(0)$. ◻

Theorem 3.5. Let X be a BCK-algebra. Then every fuzzy strong ideal of X is constant.

Proof. Let f be a fuzzy strong ideal of X . Since X is a BCK-algebra, $0 * a = 0$ for all $a \in X$. Using (FS_3) with $x = 0$, we get

$$f(a) \geq \min\{f(0), f(0 * a)\} = \min\{f(0), f(0)\} = f(0).$$

By (FS_1) , $f(0) \geq f(a)$. Hence $f(a) = f(0)$ for all $a \in X$. Therefore f is constant. ◻

Remark 3.6. The previous theorem shows that fuzzy strong ideals are non-trivial mainly in proper BCI-algebras. In BCK-algebras, the strong condition forces constant fuzzy subsets.

4. Level Characterization

Theorem 4.1. Let X be a BCI-algebra and let $f: X \rightarrow [0,1]$ be a fuzzy subset. Then f is a fuzzy strong ideal of X if and only if every non-empty upper level subset $U(f, t)$ is a strong ideal of X .

Proof. Suppose first that f is a fuzzy strong ideal. Since f is a fuzzy ideal, every non-empty upper level subset $U(f, t)$ is an ideal of X .

Let $x \in U(f, t)$ and suppose $x * a \in U(f, t)$. Then $f(x) \geq t$ and $f(x * a) \geq t$. By (FS_3) ,

$$f(a) \geq \min\{f(x), f(x * a)\} \geq t.$$

Thus $a \in U(f, t)$. Hence $U(f, t)$ is a strong ideal.

Conversely, suppose every non-empty upper level subset $U(f, t)$ is a strong ideal. Since every strong ideal is an ideal, every non-empty upper level subset is an ideal. Hence f is a fuzzy ideal.

Now let $x, a \in X$ and put

$$t = \min\{f(x), f(x * a)\}.$$

Then $x \in U(f, t)$ and $x * a \in U(f, t)$. Since $U(f, t)$ is a strong ideal, we get $a \in U(f, t)$. Thus

$$f(a) \geq t = \min\{f(x), f(x * a)\}.$$

Therefore f satisfies (FS_3) . Hence f is a fuzzy strong ideal. $\square$

Corollary 4.2. Let X be a BCI-algebra and let f be a fuzzy strong ideal of X . Then

$$A_f = \{x \in X: f(x) = f(0)\}$$

is a strong ideal of X .

Proof. By (FS_1) , $f(0)$ is the greatest value of f . Hence $A_f = U(f, f(0))$. By Theorem 4.1, $U(f, f(0))$ is a strong ideal. Therefore A_f is a strong ideal. $\square$

Corollary 4.3. Let X be a BCI-algebra, let $f: X \rightarrow [0,1]$ be a finite-valued fuzzy subset, and suppose

$$f(X) = \{t_1, t_2, \dots, t_n\}, \quad t_1 > t_2 > \dots > t_n.$$

Then f is a fuzzy strong ideal of X if and only if each upper level subset $U(f, t_i)$, $1 \leq i \leq n$, is a strong ideal of X .

Proof. Suppose first that f is a fuzzy strong ideal of X . By Theorem 4.1, every non-empty upper level subset of f is a strong ideal; in particular, each $U(f, t_i)$, $1 \leq i \leq n$, is a strong ideal of X .

Conversely, assume that every $U(f, t_i)$, $1 \leq i \leq n$, is a strong ideal of X . Let $t \in [0,1]$ and suppose $U(f, t)$ is non-empty. If $t \leq t_1$, then $U(f, t) = X = U(f, t_1)$. Otherwise, since $t_1 > t_2 > \dots > t_n$, there is an index $i \in \{1, \dots, n-1\}$ such that $t_i \geq t > t_{i+1}$. Because f takes no values strictly between t_i and t_{i+1} , we have $U(f, t) = U(f, t_i)$.

Thus every non-empty upper level subset $U(f, t)$ is one of the strong ideals $U(f, t_i)$. Theorem 4.1 now implies that f is a fuzzy strong ideal of X . $\square$

5. Characteristic Functions

Definition 5.1. [3,5] Let I be a non-empty subset of X . Define the characteristic fuzzy subset f_I of I by

$$f_I(x) = \begin{cases} 1, & x \in I, \\ 0, & x \notin I. \end{cases}$$

Theorem 5.2. Let X be a BCI-algebra and let I be a non-empty subset of X . Then I is a strong ideal of X if and only if f_I is a fuzzy strong ideal of X .

Proof. Suppose first that I is a strong ideal. The non-empty upper level subsets of f_I are I and X . Since I and X are strong ideals, Theorem 4.1 implies that f_I is a fuzzy strong ideal.

Conversely, suppose f_I is a fuzzy strong ideal. Since $I = U(f_I, 1)$, Theorem 4.1 implies that I is a strong ideal. ▢

Theorem 5.3. Let X be a BCI-algebra and let I be a strong ideal of X . Let $\alpha, \beta \in [0,1]$ with $\alpha > \beta$. Define $f: X \rightarrow [0,1]$ by

$$f(x) = \begin{cases} \alpha, & x \in I, \\ \beta, & x \notin I. \end{cases}$$

Then f is a fuzzy strong ideal of X .

Proof. The non-empty upper level subsets of f are I and X . Since I and X are strong ideals, Theorem 4.1 implies that f is a fuzzy strong ideal. ▢

6. Homomorphic Preimages

Definition 6.1. [2] Let X and Y be BCI-algebras. A map $h: X \rightarrow Y$ is called a BCI-homomorphism if

$$h(x * y) = h(x) * h(y), \quad \forall x, y \in X,$$

and $h(0) = 0$.

Theorem 6.2. Let $h: X \rightarrow Y$ be a BCI-homomorphism and let f be a fuzzy strong ideal of Y . Define $g: X \rightarrow [0,1]$ by

$$g(x) = f(h(x)), \text{ for all } x \in X.$$

Then g is a fuzzy strong ideal of X .

Proof. First, $g(0) = f(h(0)) = f(0)$. Since f is a fuzzy ideal, $f(0) \geq f(h(x))$ for all $x \in X$. Hence $g(0) \geq g(x)$.

For $x, y \in X$,

$$\begin{aligned} g(x) &= f(h(x)) \geq \min\{f(h(x) * h(y)), f(h(y))\} = \min\{f(h(x * y)), f(h(y))\} \\ &= \min\{g(x * y), g(y)\}. \end{aligned}$$

Thus g is a fuzzy ideal.

Now let $x, a \in X$. Since f is fuzzy strong,

$$f(h(a)) \geq \min\{f(h(x)), f(h(x) * h(a))\}.$$

Since $h(x) * h(a) = h(x * a)$, we get

$$g(a) \geq \min\{g(x), g(x * a)\}, \text{ for all } x, a \in X.$$

Thus g is a fuzzy strong ideal of X . ▢

7. Meet-Semilattice Property

Let $FSI(X)$ denote the family of all fuzzy strong ideals of X . For $f, g \in FSI(X)$, define $f \wedge g$ by

$$(f \wedge g)(x) = \min\{f(x), g(x)\}, \quad \forall x \in X.$$

Theorem 7.1. If f and g are fuzzy strong ideals of X , then $f \wedge g$ is a fuzzy strong ideal of X .

Proof. Put $p = f \wedge g$. We verify (FS1)-(FS3) directly.

$$p(0) = \min\{f(0), g(0)\} \geq \min\{f(x), g(x)\} = p(x), \text{ for all } x \in X.$$

For $x, y \in X$, the fuzzy-ideal inequalities for f and g give

$$\begin{aligned} p(x) &= \min\{f(x), g(x)\} \geq \min\{f(x * y), f(y), g(x * y), g(y)\} \\ &= \min\{p(x * y), p(y)\}. \end{aligned}$$

Thus (FS2) holds. Now let $x, a \in X$. Because f and g satisfy (FS3),

$$f(a) \geq \min\{f(x), f(x * a)\} \text{ and } g(a) \geq \min\{g(x), g(x * a)\}.$$

Taking the minimum of the two inequalities gives

$$p(a) = \min\{f(a), g(a)\} \geq \min\{f(x), f(x * a), g(x), g(x * a)\} = \min\{p(x), p(x * a)\}.$$

Therefore p satisfies (FS1), (FS2), and (FS3) for all relevant elements, so p is a fuzzy strong ideal.

Corollary 7.2. The family $\text{FSI}(X)$, ordered pointwise by $f \leq g$ if and only if $f(x) \leq g(x)$ for every $x \in X$, is a meet-semilattice.

8. Examples

Example 8.1. Let $X = \{0, a, b\}$ and define $*$ by the following table:

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

This is a BCI-algebra, but it is not a BCK-algebra because $0 * a = b \neq 0$.

Let $I = \{0\}$. Then I is a strong ideal. Indeed, if $x \in I$ and $x * y \in I$, then $x = 0$. From the table, $0 * y = 0$ only when $y = 0$. Hence $y \in I$.

Define $f: X \rightarrow [0,1]$ by

$$f(0) = 0.8, \quad f(a) = 0.3, \quad f(b) = 0.3.$$

The non-empty upper level subsets of f are $\{0\}$ and X . Both are strong ideals. Hence f is a fuzzy strong ideal.

Example 8.2. Let $X = \{0, a, b, 1\}$ be the four-element Boolean BCK-algebra. With a and b as complementary atoms, $x * y = x \wedge y'$; equivalently, the operation is given by the following Cayley table:

$*$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

$$f(0) = 0.9, \quad f(a) = 0.9, \quad f(b) = 0.4, \quad f(1) = 0.4.$$

The upper level subsets are $\{0, a\}$ and X , which are ideals. Thus f is a fuzzy ideal. However, f is not constant. Since X is a BCK-algebra, Theorem 3.5 shows that f is not a fuzzy strong ideal.

This example shows that a fuzzy ideal need not be a fuzzy strong ideal.

Example 8.3. Let $X = \{0, a, b\}$ and let

$$\begin{aligned} f(0) &= 0.8, & f(a) &= 0.3, & f(b) &= 0.3, \\ g(0) &= 0.9, & g(a) &= 0.4, & g(b) &= 0.4. \end{aligned}$$

Both f and g are fuzzy strong ideals. Their meet $p = f \wedge g$ is

$$p(0) = 0.8, \quad p(a) = 0.3, \quad p(b) = 0.3.$$

Thus $p = f$, and hence p is a fuzzy strong ideal.

9. Conclusion

In this paper, we introduced fuzzy strong ideals in BCI-algebras. The definition was obtained by applying the level subset method to the classical strong ideal condition.

The principal results are the upper level characterization; the fact that $U(f, f(0))$ is a strong ideal; the constancy of fuzzy strong ideals on BCK-algebras; the characterization of strong ideals by two-valued characteristic fuzzy subsets; preservation under

homomorphic preimages; and closure of the family of fuzzy strong ideals under pointwise meet.

Future work may study anti-fuzzy strong ideals, fuzzy strong P-ideals, fuzzy strong H-ideals and fuzzy strong filters.

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